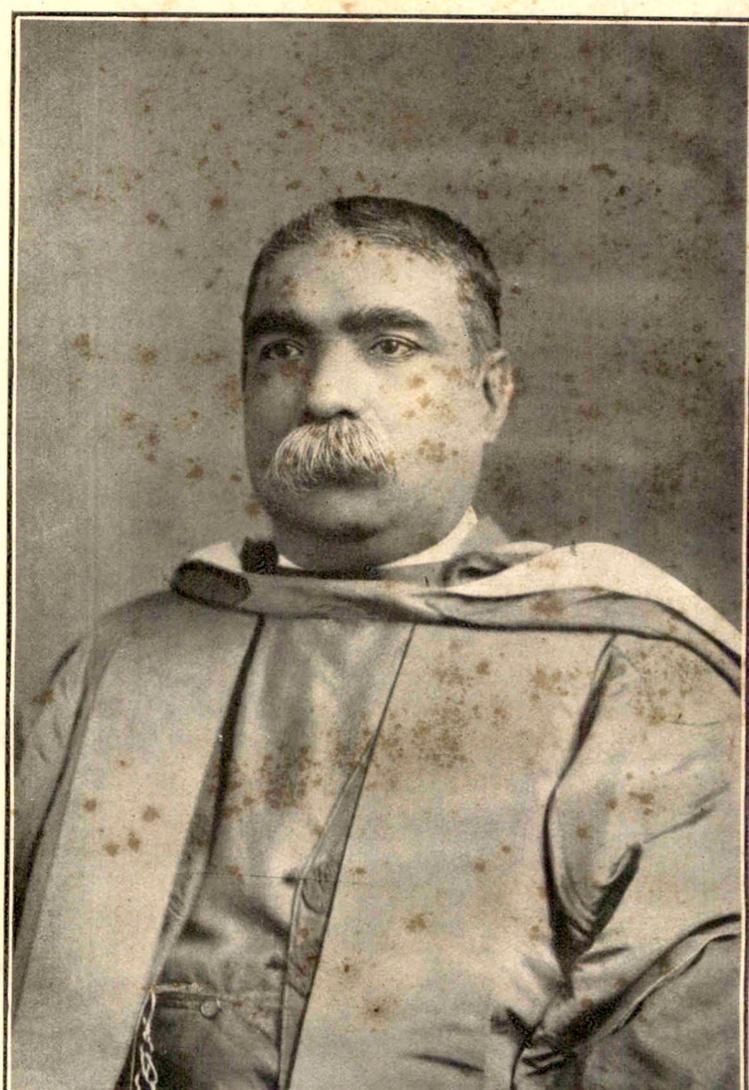


SIR ASUTOSH MUKHOPADHYAY,

President of the Calcutta Mathematical Society (1908—1924).

PRINTED BY BHUPENDRALAL BANERJEE
CALCUTTA UNIVERSITY PRESS, SENATE HOUSE, CALCUTTA.

Reg. No. 405B, November, 1930—650.

COMMEMORATION VOLUME

ON THE OCCASION OF THE

TWENTIETH ANNIVERSARY

OF THE FOUNDATION OF

THE CALCUTTA MATHEMATICAL SOCIETY

IN 1928

Bulletin

Vol. XX

With two plates, two sheets of diagrams, and 4 tables and 13 figures in the text.

CALCUTTA UNIVERSITY PRESS

1930

PREFACE

At a meeting of the Calcutta Mathematical Society, held on the 29th Januar 8, it was resolved that, in connection with the 20th anniv y of the foundation of the Society, a volume of the Bulletin, to be called the Commemoration Volume, should be brought out. It was further resolved that the volume should consist of papers contributed by Honorary Members of the Society and other eminent mathematicians and that the President should be authorized to invite contributions.

In pursuance of the aforesaid resolutions, nearly forty mathematicians were requested to contribute to the Commemoration Volume, and it is gratifying to note that the response to the request was very satisfactory. The volume that is being sent out to the world, contains, in addition to the Presidential Address, 26 papers: of these papers America contributes 2, Austria 1, France 1, Germany 4, Great Britain 6, Hungary 2, India 4, Italy 1, Japan 2, Poland 1, Russia 1, Switzerland 1. To all those who responded to my request and specially to those whose contributions appear in this volume, I take this opportunity to express, on behalf of the Society, sincere thanks. The delay in the final printing off of many of the papers has been considerable, chiefly due to the unavoidable delay because of the correction of proof-sheets across thousands of miles. In some cases the proofs had to be corrected more than once by the authors living far away in Europe. Some delay is inevitable in connection with publications of an international character; but, if this Commemoration Volume has been delayed longer in appearance than was necessary, I offer an apology to the contributors, as, in spite of my numerous preoccupations, I had the sole charge of the work of editing the volume.

The deepest gratitude of the Society is due to the Syndicate of the Calcutta University and to the Press and Publication Committee for the uniform kindness which they have shown in meeting the

wishes of the Society in every matter connected with the printing and binding of the Commemoration Volume. I also record my appreciation of the efforts of the officials of the Calcutta University Press to cope with the work of printing the volume, consisting, as it does, of papers not only in English but also in French, German and Italian.

CALCUTTA.
The 23rd October, 1390.

GANESH PRASAD,
President,
Calcutta Mathematical Society.

CONTENTS

THE TRANSMISSION OF FREE ELECTRIC WAVES IN THE ATMOSPHERE

BY

SIR JOSEPH LARMOR (*Cambridge*)

[*Read July 1, 1928*]

It has been explained, *Philosophical Magazine*, Dec. 1924, that the presence of free electrons or ions in the auroral region of the upper atmosphere can give an adequate account of the now familiar bending of wireless signals so as to keep parallel to the earth's surface

A limited region containing a substantial number of ions of various kinds has a low refractive index μ, which may be less than unity, given by

$$\mu^2 = \frac{c^2}{c'^2} = K - \Sigma \frac{4\pi N_r e_r^2}{p^2 m_r}, \quad \frac{2\pi}{p} = \frac{\lambda}{c},$$

where K depends on density, being 1·006 for standard air; so the rays may be deflected by bending downward which may even become complete gradual reflection, but yet with only slight absorption when the ionic free paths are long. For wireless waves of the lengths ordinarily in use, the index of the medium would fall even towards zero for quite moderate numerical densities of the electrons, if other causes did not intervene: for a wavelength of one kilometre the required density is only about 9.10^4, increasing for shorter waves as the inverse square of their length. If there exists a stratum aloft of such electron-density, waves longer than a kilometre cannot escape into space, but all except the slight proportion travelling nearly vertically must be bent back to the earth before they get anywhere near it. In the greater freedom of the rarified upper atmosphere ionization would however be dissipated more rapidly, unless it is continually renewed

Local regions strongly ionized and comparable in extent to the wave-length would act as obstacles to transmission, even as total reflectors turning back and dispersing the waves. in this respect short waves must be at a disadvantage just as is blue light for penetration through the sky. But now-a-days reception is so sensitive that such attenuation is unimportant, unless it is accompanied by change of velocity.

The circumstances of atmospheric electric propagation parallel to the earth, and also those here noted, restricting escape of radiation into space, will be affected sensibly, as E. V. Appleton has remarked, by the earth's magnetic field, but in complicated ways, that will now be illustrated by estimates for some of the simpler cases.

A wave-train in an ionised medium travelling in the direction of a magnetic field.

The equations of motion of an ion transverse to z, the direction of propagation of the waves, are, in a magnetic field H_0 along z,

$$m\ddot{x}=eP+eH_0\dot{y}$$

$$m\ddot{y}=eQ-eH_0\dot{x}$$

equivalent together to

$$m\ddot{\xi}=e(P+iQ)-ieH_0\dot{\xi}, \qquad \xi=x+iy$$

The equations of the electrodynamic field of the radiation, with N electrons per unit volume (P,Q,R), being the electric and (α,β,γ) the magnetic field, all functions of $t\text{-}z/c'$ alone, become when R,γ are taken to be null

$$-\frac{\partial\beta}{\partial z}=Kc^{-2}\dot{P}+4\pi Ne\dot{x} \qquad -\frac{\partial Q}{\partial z}=-\frac{\partial\alpha}{\partial t}$$

$$\frac{\partial\alpha}{\partial z}=Kc^{-2}\dot{Q}+4\pi Ne\dot{y} \qquad \frac{\partial P}{\partial z}=-\frac{\partial\beta}{\partial t}$$

$$0=Kc^{-2}\dot{R}+4\pi Ne\dot{z} \qquad 0=-\frac{\partial\gamma}{\partial t}$$

thus requiring also $\dot{z}$ to be null, so that the waves travelling along the direction of z are wholly transverse,

Thus

$$\frac{\partial}{\partial z}(a+\iota\beta)=-\iota Kc^{-1}\frac{\partial}{\partial t}(P+\iota Q)-\iota 4\pi Ne\frac{\partial}{\partial t}(x+\iota y)$$

$$\frac{\partial}{\partial z}(P+\iota Q)=\iota\frac{\partial}{\partial t}(a+\iota\beta)$$

so that, eliminating $a+i\beta$,

$$\frac{\partial^2}{\partial z^2}(P+\iota Q)=Kc^{-2}\frac{\partial^2}{\partial t^2}(P+\iota Q)+4\pi Ne\frac{\partial^2}{\partial t^2}(x+\iota y)$$

which is to be combined with the equation of movement of free ions

$$m\frac{\partial^2}{\partial t^2}(x+\iota y)=e(P+\iota Q)-\iota eH_0\frac{\partial}{\partial t}(x+\iota y).$$

The system is thus cyclic, as its relations are consolidated into two equations with complex variables, equivalent to four when real and imaginary parts are separated.

Consider a harmonic wave-train expressed by

$$P+\iota Q=F_0 e^{\iota pt}e^{\iota nz} \qquad a+\iota\beta=-\iota\frac{n}{p}F_0 e^{\iota pt}e^{\iota nz}$$

equivalent to

$$P=F_0\cos(pt+nz) \qquad a=\frac{n}{p}F_0\sin(pt+nz)$$

$$Q=F_0\sin(pt+nz) \qquad \beta=-\frac{n}{p}F_0\cos(pt+nz)$$

At each instant its vector field (P,Q) has the form of a right-handed spiral staircase with axis along the train, and the field (a,β) is a conjugate one at right angles, being a quarter period in advance in phase, the configuration of these fields travelling inward towards the origin with velocity p/n.

Each ion describes a circle, with vector radius along the electric vector, involving $x+iy$, and with real coefficients, being determined by its equation of vibration as above.

On substitution, c' representing p/n the velocity of propagation of phase and ξ representing as before $x+iy$,

$$\left(\frac{1}{c'^2}-\frac{K}{c^2}\right)F_0=4\pi Ne\xi$$

$$eF_0=-(mp^2+eH_0p)\xi$$

Thus

$$\frac{c^2}{c'^2}=K-\frac{4\pi Ne^2c^2}{mp^2+eH_0p}$$

For c' to be real, so that there may be a chance for rays to be transmitted instead of being turned aside, the right hand side must be positive. Obviously this will be precluded if the period $2\pi/p$ is too great, whatever the imposed field H_0 may be. Thus all waves of length exceeding $(\pi m/e^2N)^{\frac{1}{2}}$ or roughly $3.10^6N^{\frac{1}{2}}$ would be turned back to the earth before they reach a stratum of electrons of numerical density N if such exist : it is only shorter waves that will have a chance to escape upward into space.

Also, the value of the denominator, as affected by the magnetic field, is roughly, when the mass m is that of an electron, $10^{-27}p^2+10^{-20}H_0p$: thus, whatever N may be, the circumstances become critical, for propagation in the direction of H_0, when p falls towards the value 10^7H_0, or in the earth's magnetic field when the standard wavelength rises towards a quarter of a kilometre—beyond that limit the medium transmits only one component and so circularly polarizes the radiation, irrespective of the number of electrons the medium contains. The reason is that the motion of each individual electron would increase without limit as this period is approached. unless inhibited by other causes.

When N is given, the precise condition for c'^2 to be positive is that

$$mp^2+eH_0p \quad \text{exceed} \quad 4\pi K^{-1}Ne^2c^2.$$

Thus the period $2\pi/p$ being positive, waves cannot penetrate against the direction of an imposed magnetic field H_0 if p is less than

$$-\frac{1}{2}\frac{e}{m}H_0+\left(\frac{1}{4}\frac{e^2}{m^2}H_0^2+\frac{4\pi}{Km}Ne^2c^2\right)^{\frac{1}{2}};$$

for the opposite direction of travel the sign of H_0 would be reversed in this formula.

In the absence of a magnetic field the lower limit of wavelength for penetration, in any direction, is a kilometre when N is about 10^3. The influence

of a magnetic field on it may be considerable, and is inversely as N, as comparison of the terms under the radical shows: when N/H_0^2 is $\frac{1}{4}10^5$ the two terms under the radical are of the same order.

A plane-polarized ray travelling along the field H_0 would have its plane of polarization rotated. The difference of phase, measured as length, of its two circularly polarized components, would be the difference of the values of $\int \mu ds$: and by the Fermat principle that this integral is stationary in value as regards change to all adjacent ray paths, this difference of values can be estimated by integrations made for both along the same path, if the actual paths are not far apart. The values of μ^2 which is c^2/c'^2 are given above for the two rays, differing according to the sign of H_0. For periods well below the critical value $10^7 H_0$ of p, the difference of the indices for the two signs of H_0 is approximately $4\pi \frac{Ne}{K^{\frac{1}{2}}} \frac{e^2}{m^2} \frac{H_0}{p^3} c^2$: and this multiplied by length of path l and by $pc/K^{\frac{1}{2}}$ will give for the angle of rotation of the plane of polarization in travelling that path the value $Nel \frac{c^2}{m^2} \frac{H_0 \lambda^3}{2\pi^2 c}$, which is $8.10^{-13} H_0 N \lambda^3$ per cm. and increases rapidly with λ . taking H_0 to be $\frac{1}{5}$ it is 2.10^{-7}N for waves of 50 metres. It is to be noted that if this rotation could be observed, it would determine an average value of N along the path of the ray: that of dN/dh is determined by the curvature of the earth.

Actually it appears (*loc. cit.*) that a vertical atmospheric gradient dN/dh of only a few electrons per kilometre is required to carry the ray round the earth's curvature. In such a case N may be so small that the square root can be expanded: so that the limits of p for absence of penetration are from $18N/KH_0$, which may be as low as 10^2, down to zero, corresponding to impracticable periods of order of more than one-tenth of a second.

But though its gradient at the level of the rays is thus fixed, N itself may be large. For example, if N were of the order of 10^6, and H_0 is taken as $\frac{1}{6}$ which is of the order of the earth's field, the square root term would be about 7, so that the range of p for absence of transmission * would be up to $6.10^7 H_0$

* An application to the Sun's atmosphere presents itself. It was found by G. E. Hale and his associates (*Astrophys. J.* 1918) that the general magnetic field of the Sun is rapidly extinguished with height in his atmosphere. Quite a small density of free electrons or ions, provided their free paths are long enough, can prevent any electromagnetic changes, if not too abrupt, from being transmitted across to the outside. Especially is this so when the atmosphere is in motion: the axial rotation of an atmosphere can shield off all but the meridianal components of a star's internal magnetic field: *cf. Phil. Mag.*, January, 1884.

for propagation against the direction of H_0, and up to $8.10^7 H_0$ in its own direction corresponding to wave-lengths down to 300 metres and to 400 metres. Thus the actual transmission of shorter waves puts a superior limit to the number of electrons that can be present in the strata concerned with it.

Imposed magnetic field oblique to wave fronts.

For steady propagation of waves travelling oblique to H_0 the conditions are much more complex: the planes of the magnetic and electric vibrations are not along the fronts of the waves, each has a longitudinal component, the two components combining as will appear into an elliptic vibration. When such waves travel into a region of smaller magnetic field, or one differently directed, the circumstances no longer fit together, and, perhaps on this account as well as from change of velocity, the ray may tend to turn round horizontally; other causes also, not however feasibly a horizontal gradient of N, may give rise to horizontal curvature of the rays. On both grounds the usual determinations of the direction of the source may be deceptive. The general problem of propagation in a non-uniform magnetic field is perhaps intractable; but if the velocities of transmission are tabulated for different magnitudes and relative directions of uniform field, the Fermat principle of minimal time may lead by graphical procedure to some notion of the actual ray paths.

These considerations may be further elucidated by working out a simple case. If the imposed magnetic field $(0, \beta_0, 0)$ is parallel to the field $(0, \beta, 0)$ of the radiation, it may be presumed that the electric field will be transverse to β and that the polarisation will not rotate.* Let us explore therefore the scheme expressed by a field of radiation $(0, \beta, 0)$ and (P, O, R). All quantities are functions of $pt-nz$ alone. We have therefore, now writing N_0 for the density of electrons,

$$-\frac{\partial \beta}{\partial z} = 4\pi u = Kc^{-2}\dot{P} + 4\pi N_0 e\dot{x}$$

$$0 = \frac{\partial \beta}{\partial x} = 4\pi w = Kc^{-2}\dot{R} + 4\pi N_0 e\dot{z}$$

* In the complementary type of polarization, electric vibration along the imposed magnetic field, the latter will obviously be inoperative on the oscillating ions. The formulae for the velocities of the two cases here worked out have been recorded already by E. V. Appleton and M. A. F. Barnett, *Proc. Camb. Phil. Soc.*, March, 1925, as the writer has been reminded by an early draft of a general discussion by S. Goldstein.

while

$$-\frac{\partial \beta}{\partial t}=\frac{\partial \mathrm{P}}{\partial z}\ .$$

For a free ion at (x, y, z)

$$m\ddot{x}=e\mathrm{P}-e\beta_0\dot{z},\ m\ddot{y}=0,\ m\ddot{z}=e\mathrm{R}+e\beta_0\dot{x},$$

in which for periodic waves $\partial/\partial t$ is ip: thus

$$\dot{x}=\frac{-e\dot{\mathrm{P}}+\frac{e}{m}\beta_0\mathrm{R}}{mp^2+\frac{e^2}{m}\beta_0^2},\ \dot{z}=\frac{-e\dot{\mathrm{R}}-\frac{e}{m}\beta_0\mathrm{P}}{mp^2+\frac{e^2}{m}\beta_0^2}$$

Hence

$$-\frac{\partial \beta}{\partial z}=\mathrm{L}\dot{\mathrm{P}}+\mathrm{NR} \qquad \mathrm{L}=\frac{\mathrm{K}}{c^2}-\frac{4\pi \mathrm{N}_0 e^2 m}{m^2p^2+e^2\beta_0^2}$$

$$0=\frac{\partial \beta}{\partial x}=\mathrm{L}\dot{\mathrm{R}}-\mathrm{NP} \qquad \mathrm{N}=\frac{4\pi \mathrm{N}_0 e^2\beta_0}{m^2p^2+e^2\beta_0^2}$$

giving, if L′ represent $\mathrm{L}\,\partial/\partial t$,

$$(\mathrm{L}'^2+\mathrm{N}^2)\mathrm{P}=-\mathrm{L}'\ \frac{\partial \beta}{\partial z},\quad (\mathrm{L}'^2+\mathrm{N}^2)\mathrm{R}=-\mathrm{N}\ \frac{\partial \beta}{\partial z}$$

while as above

$$-\frac{\partial \beta}{\partial t}=\frac{\partial \mathrm{P}}{\partial z}.$$

Thus

$$\frac{\partial^2 \beta}{\partial z^2}=\left(\mathrm{L}-\frac{\mathrm{N}^2}{\mathrm{L}p^2}\right)\frac{\partial^2 \beta}{\partial t^2}$$

giving

$$c'^2 = \frac{n^2}{p^2} = L - \frac{N^2}{Lp^2}$$

reducing to the value previously found for c^2/c'^2 when β_0 is null.

To permit penetration p^{-2} must exceed L^2/N^2, leading to a discriminating cubic equation in p Reversal of the direction of β_0 is without influence. for change of sign of β_0 changes only the sign of N whereas c'^{-2} involves N^2.

The longitudinal component of the electric field is in quadrantal phase to the transverse, with amplitude in ratio N/Lp, so that this field vibrates elliptically.

The general analysis to determine speed of propagation and type of vibration when the magnetic field is inclined to the front involves complex formulæ : moreover in the terrestrial application the inclination changes as the ray progresses. The most convenient expression of the results would probably be in tabular arithmetic form. The changing velocity involves horizontal bending of the ray, so that from this cause error may arise in directional findings as regards position of the sources. Perhaps the feasible way of dealing with that problem is, as remarked above, to try to lay off graphically the quickest paths, which are the rays, in the field of propagation with the local velocities marked all over it.

ST. JOHN'S COLLEGE, CAMBRIDGE.

May 9, 1928

ON THE FLOW OF A COMPRESSIBLE FLUID PAST AN OBSTACLE

BY

HORACE LAMB (*Cambridge*)

[*Read July 1, 1928*]

Mr. Glauert in a recent paper * on the two-dimensional flow of a frictionless compressible fluid past an obstacle has found the interesting result that the lift due to circulation is given by exactly the same formula as in the case of incompressibility, provided the velocity of the stream is less than the velocity of sound in the undisturbed fluid.

The flow past a *circular* cylinder, without circulation of the fluid round it, was discussed some years ago by Rayleigh † by a method of successive approximation. Considering the pressures on the surface he found that the resultant force on the cylinder was zero, as must evidently be the case when the configuration of the stream-lines is symmetrical. I have thought it worth while to extend his calculations by including circulation ; but I wish in the first place to show how his method can be used to verify Mr. Glauert's result, so far as the approximation holds.

It is assumed that the stream is uniform at a great distance from the obstacle, and the motion therefore everywhere irrotational. It is not necessary to make any special assumption as to the relation between the pressure p and the density ρ, although the adiabatic hypothesis is the most natural. Writing

$$c^2 = dp/d\rho, \qquad \dots \quad (1)$$

* *Proc. Roy. Soc.*, Vol. 118, p. 113 (1928).

† *Phil. Mag.*, Vol. 32, p. 1 (1916) ; *Sc. Papers*, Vol. 6, p. 402.

the dynamical equations reduce to

$$c^2\frac{d\rho}{\rho}=-\frac{1}{2}\,d(q^2), \qquad \dots \quad (2)$$

where q is the fluid velocity. The equation of continuity may therefore be written (after Rayleigh)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{1}{2c^2}\left\{\frac{\partial(q^2)}{\partial x}\,\frac{\partial\phi}{\partial x}+\frac{\partial(q^2)}{\partial y}\,\frac{\partial\phi}{\partial y}\right\}, \qquad \dots \quad (3)$$

or, in polar co-ordinates,

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\,\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2\phi}{\partial\theta^2}=\frac{1}{2c^2}\left\{\frac{\partial(q^2)}{\partial r}\,\frac{\partial\phi}{\partial r}+\frac{\partial(q^2)}{r\partial\theta}\,\frac{\partial\phi}{r\partial\theta}\right\}\dots \quad (4)$$

We take the origin of r in the immediate neighbourhood of the obstacle, and the initial line of θ parallel to the general direction of the stream, which is (say) horizontal, and towards the right hand.

If c (the velocity of compressional waves) were infinite the value of ϕ at a distance would tend to the form

$$\phi_1=-\mathrm{V}\,r\cos\theta+\mathrm{C}\theta \qquad \dots \quad (5)$$

where V is the velocity of the undisturbed stream, and 2πC is the circulation in the counter-clockwise sense. We adopt this as a first approximation and substitute on the right-hand side of (4). We may also, consistently, for the next approximation, replace c by its constant value at infinity (c_0). From (5) we have

$$q_1{}^2=\mathrm{V}^2+\frac{2\mathrm{VC}}{r}\sin\theta \qquad \dots \quad (6)$$

$$\frac{\partial(q_1{}^2)}{\partial r}\quad\frac{\partial\phi_1}{\partial r}+\frac{\partial(q_1{}^2)}{r\partial\theta}\quad\frac{\partial\phi_1}{r\partial\theta}=\frac{2\mathrm{V}^2\mathrm{C}}{r^2}\sin 2\theta \qquad \dots \quad (7)$$

retaining only those terms which it is necessary to consider when r is increased indefinitely. Substituting in (4), integrating, and having regard to the conditions at infinity, we have in the distant regions,

$$\phi=-\mathrm{V}\,r\cos\theta+\mathrm{C}\theta-\frac{\mathrm{V}^2\mathrm{C}}{4c_0{}^2}\sin 2\theta \qquad \dots \quad (8)$$

The "complementary" terms, here omitted, would involve only negative process of r, and would not affect the subsequent calculation of the forces. Hence, with sufficient approximation, the radial and transverse velocities are

$$-\frac{\partial\phi}{\partial r}=V\cos\theta,\; -\frac{\partial\phi}{r\partial\theta}=-V\sin\theta-\frac{C}{r}+\frac{V^2C}{2c_0^2r}\cos 2\theta, \quad \dots \quad (9)$$

whence

$$q^2=V^2+\frac{2VC}{r}\left(1-\frac{V^2}{2c_0^2}\cos 2\theta\right)\sin\theta \quad \dots \quad (10)$$

The horizontal and vertical components are

$$u=V+\frac{C}{r}\left(1-\frac{V^2}{2c_0^2}\cos 2\theta\right)\sin\theta,\; v=-\frac{C}{r}\left(1-\frac{V^2}{2c_0^2}\cos 2\theta\right)\cos\theta.\dots \quad (11)$$

These agree with Mr. Glauert's results, to our order of approximation.

The forces on the obstacle can now be inferred from the modified flow at infinity, as in the original proofs of the Kutta-Joukowsky theorem. The mass of fluid enclosed at any instant within a circle of large radius r is gaining momentum upward at the rate

$$\int_0^{2\pi}\left(-\frac{\partial\phi}{\partial r}\right)\rho v r d\theta=-\pi\rho_0 VC\left(1-\frac{V^2}{4c_0^2}\right), \quad \dots \quad (12)$$

ultimately, from (9) and (11). Again, from (2),

$$\log\frac{\rho}{\rho_0}=\int_q^V\frac{d(q^2)}{2c_0^2}=\frac{V^2-q^2}{2c_0^2}, \quad \dots \quad (13)$$

and therefore

$$\frac{\rho}{\rho_0}=1+\frac{V^2-q^2}{2c_0^2}, \quad \dots \quad (14)$$

with consistent approximation. Hence, since for large values of r, ρ tends to ρ_0, we may put

$$p=p_0+c_0^2(\rho-\rho_0)=p_0-\frac{\rho_0 VC}{r}\left(1-\frac{V^2}{2c_0^2}\cos 2\theta\right)\sin\theta. \qquad \dots \quad (15)$$

The resultant upward pressure on the aforesaid mass of fluid is therefore

$$-\int_0^{2\pi} p\sin\theta\, r d\theta=\pi\rho_0 VC\left(1+\frac{V^2}{4c_0^2}\right) \qquad \dots \quad (16)$$

Comparing with (12), the lift is given by the familiar Joukowsky formula

$$L=2\pi\rho_0 VC, \qquad \dots \quad (17)$$

with (at most) a relative error of the order $(V/c_0)^4$.

In the same way it may be shown that the drag is zero.

In one respect the above investigation is inferior to that of Mr. Glauert, in that it is only an approximation; though a reasonably good one over a considerable range of the ratio V/c_0. His analysis, on the other hand, would appear to hold for all values of this ratio, short of unity. I do not suppose that Mr. Glauert would himself press his conclusions so far, and in fact a physical reason for some limitation can be found, I think, in one of his formulæ.

He finds

$$\frac{\rho}{\rho_0}=1-\frac{\lambda}{r}(A\cos\theta+B\sin\theta), \qquad \dots \quad (18)$$

with

$$\lambda=V^2/c_0^2,\ A=0,\ B=\frac{\sqrt{(1-\lambda)}}{1-\lambda\sin^2\theta}\ \frac{C}{V} \qquad \dots \quad (19)*$$

Taking $\theta=\frac{1}{2}\pi$, we have

$$\frac{\rho}{\rho_0}=1-\frac{\lambda}{\sqrt{(1-\lambda)}}\cdot\frac{C}{Vr} \qquad \dots \quad (20)$$

* C is written, as in the rest of this paper, for Mr. Glauert's $K/2\pi$.

This would make ρ *negative* for

$$r < \frac{C}{V} \cdot \frac{\lambda}{\sqrt{(1-\lambda)}} \qquad \dots \quad (21)$$

If λ is sufficiently near to unity, this limiting value of r may be so great that the higher powers of $1/r$, omitted in (18), are still negligible.

Turning now to Rayleigh's problem of the circular cylinder, and introducing circulation, the first approximation is

$$\rho_1 = -V\left(r + \frac{a^2}{r}\right)\cos\theta + C\theta, \qquad \dots \quad (22)$$

where a is the radius. Hence

$$q_1{}^2 = V^2\left(1 - 2\frac{a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right) + 2VC\left(\frac{1}{r} + \frac{a^2}{r^3}\right)\sin\theta + \frac{C^2}{r^2}. \qquad (23)$$

From this we derive, after some reductions,

$$\frac{\partial(q_1{}^2)}{\partial r}\frac{\partial\phi_1}{\partial r} + \frac{\partial(q_1{}^2)}{r\partial\theta}\frac{\partial\phi_1}{r\partial\theta}$$

$$= \left\{V^3\left(\frac{8a^4}{r^5} - \frac{4a^6}{r^7}\right) + \frac{4VC^2}{r^3}\right\}\cos\theta$$

$$-\frac{4V^3a^2}{r^3}\cos 3\theta + V^2C\left(\frac{2}{r^2} + \frac{8a^2}{r^4} - \frac{2a^4}{r^6}\right)\sin 2\theta \quad \dots \quad (24)$$

Now the "particular integral" of the equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = r^m e^{in\theta}$$

is

$$\phi = \frac{r^{m+2}}{(m+2)^2 - n^2} e^{in\theta},$$

except when $m+2 = \pm n$, in which case we have

$$\phi = \frac{r^{m+2}\log r}{2(m+2)} e^{in\theta}.$$

Hence, substituting from (24) on the right-hand side of (4), integrating, and adding suitable complementary terms, we find

$$\left.\begin{aligned} 2c_0{}^2\phi = {} & 2c_0{}^2\left\{-V\left(r+\frac{a^2}{r}\right)\cos\theta+C\theta\right\} \\ & +\left\{\frac{A_1}{r}+V^3\left(\frac{a^4}{r^3}-\frac{a^6}{6r^5}\right)-\frac{2VC^2}{r}\log\frac{r}{a}\right\}\cos\theta \\ & +\left(\frac{A^3}{r^3}+\frac{V^3a^4}{2r}\right)\cos 3\theta \\ & +\left(\frac{B_2}{r^2}-\frac{1}{2}V^2C-\frac{2V^2Ca^2}{r^2}\log\frac{r}{a}-\frac{V^2Ca^4}{6r^4}\right)\sin 2\theta \end{aligned}\right\} \quad \ldots \quad (25)$$

The condition that $\partial\phi/\partial r=0$ for $r=a$ requires

$$A_1=\frac{13}{6}V^3a^2-2VC^2,\ A_3=-\frac{1}{6}V^3a^4,\ B_2=-\frac{2}{3}V^2Ca^2 \quad \ldots \quad (26)$$

The terms in (25) which are independent of C agree with Rayleigh's calculation.

The value of ϕ at infinity tends to the same form (8) as in the more general argument, and the inferences as to lift and drag are necessarily the same They may be verified by a calculation of the effect of the pressures on the surface of the cylinder as follows .

We have neglected the variation of c within the field.*

Hence, from (2),

$$\frac{\rho}{\rho_0}=e^{\frac{1}{2}(V^2-q^2)/c_0{}^2}, \quad \ldots \quad (27)$$

and therefore

$$p=p_0+c_0{}^2(\rho-\rho_0)=p_0-c_0{}^2\rho_0+\rho_0c_0{}^2e^{\frac{1}{2}(V^2-q^2)/c_0{}^2} \quad \ldots \quad (28)$$

* This does not mean that we are restricted to Boyle's Law. For instance, c_0 may have its "adiabatic" value.

The resultant pressure on the cylinder at right angles to the stream is therefore

$$-\int_0^{2\pi} p \sin\theta \, a d\theta = -\rho_0 c_0{}^2 a e^{\frac{1}{2}V^2/c_0{}^2} \int_0^{2\pi} e^{-\frac{1}{2}q^2/c_0{}^2} \sin\theta \, d\theta, \quad \ldots \quad (29)$$

where q has its surface value. Putting $r=a$ in (25), we have

$$-\frac{q}{c_0} = \frac{\partial\rho}{c_0 a \partial\theta} = \frac{C}{ca} + \left(\frac{2V}{c_0} + \frac{2}{3}\frac{V^3}{c_0{}^3} + \frac{VC^2}{c^3 a^2}\right)\sin\theta$$

$$-\frac{1}{2}\frac{V^3}{c_0^3}\sin 3\theta - \frac{4}{3}\frac{V^2 C}{c^3 a}\cos 2\theta, \quad \ldots \quad (30)$$

or, writing for shortness

$$\lambda = V/c_0, \; \mu = C/c_0 a, \quad \ldots \quad (31)$$

$$-\frac{q}{c_0} = \mu + \left(2\lambda + \frac{2}{3}\lambda^3 + \lambda\mu^2\right)\sin\theta - \frac{1}{2}\lambda^3 \sin 3\theta - \frac{4}{3}\lambda^2\mu \cos 2\theta$$

$$= \mu - \frac{4}{3}\lambda^2\mu + \left(2\lambda - \frac{5}{6}\lambda^3 + \lambda\mu^2\right)\sin\theta$$

$$+ \frac{8}{3}\lambda^2\mu \sin^2\theta + 2\lambda^3 \sin^3\theta, \quad \ldots \quad (32)$$

This is of the form

$$-\frac{q}{c_0} = \alpha + \beta\sin\theta + \gamma\sin^2\theta + \delta\sin^3\theta, \quad \ldots \quad (33)$$

where it is to be noticed that α and β are of the first order, γ and δ of the third, in the small quantities λ, μ. Denoting by u and v the parts of this expression which are even and odd functions, respectively, of $\sin\theta$, *viz.*,

$$u = \alpha + \gamma\sin^2\theta, \; v = \beta\sin\theta + \delta\sin^3\theta, \quad \ldots \quad (34)$$

we have, as far as terms of the fourth order,

$$e^{-\frac{1}{2}q^2/c_0^2}=1-\frac{1}{2}(u+v)^2+\frac{1}{8}(u+v)^4,$$

and therefore

$$\int_0^{2\pi} e^{-\frac{1}{2}q^2/c_0^2}\sin\theta\,d\theta=\int_0^{2\pi}(-uv+\tfrac{1}{3}u^3v+\tfrac{1}{3}uv^3)\sin\theta\,d\theta$$

$$=\pi\left(-\alpha\beta-\frac{3}{4}\alpha\delta-\frac{3}{4}\beta\gamma+\tfrac{1}{3}\alpha^3\beta+\tfrac{1}{2}\alpha\beta^3\right) \quad \ldots \quad (35)$$

Substituting

$$\alpha=\mu-\frac{4}{3}\lambda^2\mu,\ \beta=2\lambda-\frac{5}{6}\lambda^3+\lambda\mu^2,\ \gamma=\frac{8}{3}\lambda^2\mu,\ \delta=2\lambda^3 \quad \ldots \quad (36)$$

the last written expression reduces to

$$-2\pi\lambda\mu\left(1-\frac{1}{2}\lambda^2\right),\ \text{or}\ -2\pi\frac{\mathrm{VC}}{ac_0^2}\left(1-\frac{\mathrm{V}^2}{2c_0^2}\right). \quad \ldots \quad (37)$$

The formula (29) for the resultant force at right angles to the stream thus gives, as in (17),

$$2\pi\rho_0\mathrm{VC} \quad \ldots \quad (38)$$

to our order of approximation.

ZUR THEORIE DER SCHLICHTEN ABBILDUNGEN

VON

LUDWIG BIEBERBACH (*Berlin*).

[*Read July 1, 1928*]

In einer in den *Mathematischen Annalen* erscheinenden Arbeit hat Herr *Szego* gezeigt, dass die Abschnitte einer den Einheitskreis $|z| < 1$ schlicht abbildenden Potenzreihe

$$f(z) = z + a_2 z^2 + \dots\dots\dots$$

d h. also die Funktionen

$$f_n(z) = z + a_2 z^2 + \dots\dots + a_n z^n. \quad n = 2, 3 \dots\dots$$

den Kreis $|z| < \frac{1}{4}$ schlicht abbilden, dass es aber Potenzreihen $f(z)$ gibt, bei denen passende Abschnitte keinen Kreis $|z| < \rho$ mit $\rho > \frac{1}{4}$ schlicht abbilden.

Beim Nachdenken über diesen schönen Satz bin ich auf einen anderen geführt worden, der den eben genannten teilweise enthält, nämlich insoweit als in ihm die Existenz eines von allen Abschnitten schlicht abgebildeten Kreis behauptet wird Mein Satz lautet so .

Wenn $f(z) = z + a_2 z^2 + \dots\dots$ den $|z| < 1$ *schlicht abbildet, so gibt es*

eine Zahl ρ *derart, dass jede Potenzreihe*

$$g(z)=z+b_2z^2+\ldots\ldots$$

$$\textit{mit}\ |b_n|\leq|a_n|,\ n=2,3\ldots\ldots$$

den $|z|<\rho$ *schlicht abbildet.*

ρ *ist von den* a_2, $a_3\ldots\ldots$ *unabhängig. Es ist* $\rho_1<\rho\leqq\rho_2$, *wo* ρ_1 *die kleinste positive Wurzel von*

$$(25e-40)\rho^7-(59e-96)\rho^6+(36e-57)\rho^5-9\rho^4+10\rho^3-6\rho^2+7\rho-1=0$$

und ρ_2 *die reelle Wurzel von*

$$2x^3-6x^2+7x-1=0.$$

ist. Es ist $\rho_1=0{,}159\ldots\ldots$, $\rho_2=0{,}164\ldots\ldots$

Falls die Vermutung zutrifft, dass $|a_n|\leqq n$ *ist, so ist* $\rho=\rho_2=0{,}164\ldots\ldots$ *Zu jedem Kreis* $|z|<\rho$ *mit* $\rho>\rho_2$ *gehören Funktionen* $g(z)$, *die ihn nicht schlicht abbilden.*

Dafür dass $g(z)$ den $|z|<\rho$ schlicht abbildet, *d.h.* keinen Wert in diesem Kreis mehr als einmal annimmt, ist notwendig und hinreichend, dass

$$\frac{g(z_1)-g(z_2)}{z_1-z_2}=1+b_2(z_1+z_2)+\ldots\ldots b_n(z_1^{n-1}+\ldots\ldots z_2^{n-1})+\ldots$$

für $|z_1|<\rho$, $|z_2|<\rho$ von Null verschieden sei. Dies ist sicher der Fall, wenn

$$|b_2(z_1+z_2)+\cdots\ldots|<1$$

ist für alle $|z_1|<\rho$, $|z_2|<\rho$. Nun ist

$$(1)\qquad |b_2(z_1+z_2)+\ldots\ldots|<|a_2|2\rho+|a_3|3\rho^2+\ldots\ldots$$

Nun sind die $|a_k|$ die Koeffizienten einer Potenzreihe, die den $|z|<1$ schlicht abbildet. Für diese Koeffizienten sind Abschätzungen bekannt,

deren beste von Bieberbach ($n=2$), Löwner ($n=3$), Grandjot ($n=4$, 5) Littlewood (allgemeines n) herrühren.[1]) Hiernach ist

$$|a_2| \leq 2, \quad |a_3| \leq 3, \quad |a_4| < 6, \quad |a_5| < 8, \quad |a_n| < e^n$$

Also ist

$$|b_2(z_1+z_2)+\ldots\ldots| < 4\rho+9\rho^2+24\rho^3+40\rho^4+e\sum_6^\infty n^2\rho^{n-1}$$

$$=4\rho+9\rho^2+24\rho^3+40\rho^4+e\rho^5\,\frac{36(1-\rho)^2+13\rho(1-\rho)+2\rho^2}{(1-\rho)^3}.$$

Wird daher ρ so gewählt, dass

$$4\rho+9\rho^2+24\rho^3+40\rho^4+e\rho^5\,\frac{36(1-\rho)^2+13\rho(1-\rho)+2\rho^2}{(1-\rho)^3} < 1$$

ausfallt, so wird $|z| < \rho$ durch $g(z)$ schlicht abgebildet.

Wegen $\rho < 1$ ist es dasselbe zu verlangan, dass

$$(4\rho+9\rho^2+24\rho^3+40\rho^4-1)(1-\rho)^3+e\rho^5\{36(1-\rho)^2+13\rho(1-\rho)+2\rho^2\} < 0$$

ist. Es ist also die unter 1 gelegene positive Wurzel der entsprechenden Gleichung zu ermitteln. Ordnet man nach Potenzen von ρ so wird diese Gleichung

$$1+7\rho-6\rho^2+10\rho^3-9\rho^4+(36e-57)\rho^5-(59e-96)\rho^6+(25e-40)\rho^7=0.$$

Bricht man die Dezimalbruchentwickelung der Koeffizienten bei der vierten Dim ale ab, so wird die Gleichung

$$-1+7\rho-6\rho^2+10\rho^3-9\rho^4+40{,}8581\rho^5-64{,}3786\rho^6+27{,}9570\rho^7=0.$$

Ihre kleinste Wurzel liegt zwischen 0, 159 und 0,16.

[1] Vergl. Bieberbach: *Lehrbuch der Funktionentheorie.* Band II. S. 85 u. S. 91;
Löwner: Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I, *Math. Ann.* 89.
Szegö: Zur Theorie der schlichten Abbildungen. *Math. Ann.*
Hier findet sich die Grandjotsche Abschätzung für $n=5$. Ganz analog gewinnt man auch die für $n=4$ im Text angegebene.

Der Radius des grössten Kreises, der von allen $g(z)$ schlicht abgebildet wird, ist also jedenfalls nicht kleiner als dies $\rho_1=0,159\ldots$; er ist sogar sicher grösser als ρ_1, weil sich für $\rho=\rho_1$ sicher noch

$$|\, b_2(z_1+z_2)+\ldots\ldots \,| < 1$$

ergibt Es steht nämlich in einigen der Koeffizientenabschätzungen das Zeichen $<$.

Hat man die Abschätzung $|\, a_n \,| \leq n$ zur Verfugung, so sieht man analog. dass $|\, z \,| < \rho$ von allen $g(z)$ schlicht abgebildet wird, sobald

$$\rho\,\frac{4(1-\rho)^2+5\rho(1-\rho)+2\rho^2}{(1-\rho)^3}<1$$

ist, d h. sobald

$$\rho\{4(1-\rho)^2+5\rho(1-\rho)+2\rho^2\}-(1-\rho)^3<0$$

ist, d.h. sobald ρ nicht grösser ist als die reelle Wurzel von

$$2x^3-6x^2+7x-1=0$$

Für sie findet man $\rho_2=0,164\ldots$

Dass kein Kreis $|\, z \,| < \rho$ mit $\rho>\rho_2$ durch alle $g(z)$ schlicht abgebildet werden kann, sieht man so ein. Der Kreis $|\, z \,| < \rho$ wird sicher dann durch $g(z)$ nicht schlicht abgebildet wenn in ihm eine Nullstelle von $g'(z)$ liegt. Es genügt ein $g(z)$ anzugeben, derart, dass $g'(\rho_2)=0$ ist. Ein solches $g(z)$ wird durch

$$z-\sum_2^\infty n z^n$$

geliefert. Die Ableitung davon ist nämlich

$$1-\sum_2^\infty n^2 z^{n-1}=1-z\,\frac{4(1-z)^2+5z(1-z)+2z^2}{(1-z)^3}\,.$$

Nach dem früheren ist aber

$$\rho_2\,\frac{4(1-\rho_2)^2+5\rho_2(1-\rho_2)+2\rho_2^2}{(1-\rho_2)^3}=1$$

A PROPERTY OF ORDINAL NUMBERS

BY

W. Sierpinski (*Warsaw*)

(*Communicated by Professor Ganesh Prasad, July 29, 1928*)

A set of ordinal numbers Z will be called a *closed* set if it has the following property: if Z_1 is a subset of Z and δ is the least number $\geq$ than every number of Z_1, then δ belongs to Z.

The purpose of this paper is to show the following theorem.

*Theorem. The set of all the "divisors on the left" of an ordinal number is closed.**

Lemma. If ρ is the remainder of the number $\xi\eta$ and η is of second kind, then $\rho \geq \xi\omega$.

Proof. By hypothesis, there exists an ordinal number τ such that

$$(1) \qquad \xi\eta = \tau + \rho$$

Since $\rho > 0$, we have $\tau < \xi\eta$. Hence there exist two numbers ξ_1 and η_1, such that $\xi_1 < \xi$, $\eta_1 < \eta$ and

$$(2) \qquad \tau = \xi\eta_1 + \xi_1.$$

Since $\eta_1 < \eta$, it follows that there exists a number ρ_1 such that

$$(3) \qquad \eta = \eta_1 + \rho_1$$

where

$$(4) \qquad \rho_1 \geq \omega,$$

for η is of second kind.

By (3), we have

$$(5) \qquad \xi\eta = \xi\eta_1 + \xi\rho_1$$

and by (1) and (2):

$$(6) \qquad \xi\eta = \xi\eta_1 + \xi_1 + \rho.$$

* An analogous theorem on the divisors on the right is evident, for their set is finite. For the terms and theorems used in this paper, see my book "*Leçons sur les nombres transfinis*," Chap. X, Paris, 1928.

It follows from (5) and (6) that

$$\xi\rho_1 = \xi_1 + \rho$$

and, since $\xi_1 < \xi$, we have

$$\xi\rho_1 \leq \xi + \rho$$

and, by virtue of (4) :

$$\rho \geq \xi\omega$$

Thus our lemma is proved.

Now, let α denote any ordinal number, Z the set of all the divisors on the left of α, Z_1 a subset of Z and δ the smallest number $\geq$ than every number of Z_1. Suppose that δ does not belong to Z. Therefore α is not divisible on the left by δ. It follows that there exists a number $\mu > 0$ such that

$$\alpha = \delta\mu + \rho \text{ and } 0 < \rho < \delta. \tag{7}$$

Let ρ_1 be the greatest remainder of α which is $< \delta$ (such a remainder exists, for the set of all different remainders of an ordinal number is finite). By (7), we have

$$\rho \leq \rho_1. \tag{8}$$

Let ξ be any number of the set Z_1 ; hence for some η,

$$\alpha = \xi\eta \tag{9}$$

and (by virtue of the definition of δ and of the hypothesis that δ does not belong to Z, hence to Z_1), we have

$$\xi < \delta. \tag{10}$$

If η is of the first kind, it follows from (9) that ξ is a remainder of α and hence by (10) and the definition of ρ_1 :

$$\xi \leq \rho_1. \tag{11}$$

If η is of the second kind, it follows from (7), (9) and from the lemma, that

$$\rho \geq \xi\omega > \xi.$$

By virtue of (8), we have again (11).

So that (11) is satisfied by each ξ belonging to Z_1, contrary to the definition of δ (for $\rho_1 < \delta$).

Therefore δ must belong to Z.

Thus our theorem is established.

THE VARIATION OF LATITUDE

BY

SIR F. W. DYSON (*Greenwich*)

(*Read July 1, 1928*)

1. In *Phil. Trans.* 226 A. p. 267 (1927), Mr. Udny Yule has discussed Wolf's Sunspot numbers in a new and interesting manner. Instead of analysing the numbers themselves harmonically, he deals with their second differences. He points out that an accidental error in the second differences would lead to changes in amplitude and phase of successive periods of 11 years, but that the changes from year to year would be smooth and regular. He gives as an analogy the motion of a pendulum, disturbed by occasional shots by two boys with pea-shooters. In this paper I propose to apply Yule's method to the variation of latitude at Greenwich in the years 1900 to 1928. The observations are given in the following table and are exhibited graphically in Diag. 1. Those from 1900 to 1911 are derived from the International series using the figures given by Prof. Przybyllok in *A.N.* 4840-1, while those from 1912-1927 are obtained from Greenwich observations. It may be stated that the latter agree very closely with the results given by Kimura from the International series.

TABLE I.

Variation of Latitude at Greenwich. 1900—1927. (Unit ″·001.)

	·05	·15	·25	·35	·45	·55	·65	·75	·85	·95
1900	+ 60	+ 50	+ 10	− 10	− 40	− 50	− 60	− 60	− 70	− 20
1	− 0	+ 40	+ 70	+ 110	+ 130	+ 100	+ 60	+ 10	− 60	− 90
2	−100	− 60	− 10	+ 90	+ 160	+ 210	+ 170	+ 90	0	− 100
3	−150	−160	−100	− 20	+ 80	+ 150	+ 210	+ 190	+ 120	+ 20
4	−100	−160	−170	− 120	− 40	+ 50	+ 120	+ 170	+ 150	+ 120
5	+ 20	− 80	−120	− 140	− 100	− 40	+ 50	+ 100	+ 120	+ 140

TABLE I—*contd.*

	·05	·15	·25	·35	·45	·55	·65	·75	·85	·95
6	+ 50	− 10	− 50	− 100	− 110	− 100	− 60	− 20	+ 20	+ 40
7	+ 60	+ 50	+ 60	+ 50	+ 10	− 10	− 50	− 120	− 180	− 80
8	− 30	+ 70	+140	+ 210	+ 220	+ 180	+ 90	− 40	− 150	− 240
9	−280	−200	− 80	+ 90	+ 240	+ 320	+ 300	+ 200	+ 20	− 180
10	−210	−280	−280	− 160	+ 20	+ 190	+ 290	+ 310	+ 250	+ 120
11	− 40	−170	−240	− 190	− 140	0	+ 120	+ 183	+ 238	+ 240
12	+132	+ 4	−156	− 242	− 246	− 200	− 142	− 80	− 22	+ 40
13	+102	+160	+162	+ 110	+ 60	+ 5	− 50	− 108	− 166	− 185
14	−124	− 42	+ 28	+ 90	+ 136	+ 134	+ 87	+ 18	− 60	− 144
15	−216	−283	−170	− 55	+ 87	+ 250	+ 320	+ 178	+ 88	− 98
16	− 208	−266	−255	− 172	− 20	+ 131	+ 222	+ 250	+ 204	+ 60
17	−122	−228	−254	− 221	− 103	+ 45	+ 139	+ 183	+ 185	+ 147
18	+ 65	− 27	−116	− 180	− 186	− 62	− 1	+ 48	+ 80	+ 108
19	+111	+ 32	− 68	− 90	− 38	+ 39	+ 48	+ 1	− 53	− 66
20	− 51	− 24	+ 19	+ 75	+ 186	+ 184	+ 166	+ 105	− 30	− 120
21	−145	−128	− 83	− 2	+ 106	+ 163	+ 170	+ 118	− 23	− 133
22	−164	−166	−145	− 85	+ 23	+ 175	+ 288	+ 180	+ 30	− 70
23	−149	−206	−248	− 247	− 161	+ 24	+ 144	+ 153	+ 75	− 18
24	− 89	−143	−186	− 188	− 158	− 96	− 13	+ 27	+ 36	+ 23
25	− 1	− 31	− 77	− 84	− 69	− 19	+ 28	+ 30	− 6	− 30
26	− 43	− 49	− 50	− 44	− 31	− 5	+ 25	+ 48	+ 62	+ 69
1927	+ 69	+ 53	+ 27	+ 14	+ 17	+ 43	+ 67	+ 56	+ 22	− 27

2. The variation of latitude consists in the main of a free period of about 430 days and a forced annual term. The annual term arises probably from meteorological causes. If the forces from which this arises were perfectly regular, the forced period would be constant, and no change would be produced in the free period. As the free period goes through 6 cycles very approximately in 7 years, it may be obtained by taking the mean of 7 successive years. For the years 1900—1911 it was taken as + ″·084 ×

$\cos(36°t+117°)$ where t is measured in tenths of a year (*M.N.*, *R.A.S.*, Vol. 78, p. 452). This term was taken out of the figures in Table 1 for each tenth of a year from 1900 to 1911. Following Mr. Yule, equations of the form $u_{t+2}=pu_{t+1}+qu_t\pm\epsilon$ hold, where t has successive values from 1 to 108. These were solved by least squares to find p and q, leading to the values $p=+1\cdot68$; $q=-\cdot950$.

Thus the free period is given by the difference equation

$$u_{t+2}=1\cdot68\,u_{t+1}-\cdot950\,u_t.$$

Solving the quadratic $x^2=1\cdot68\,x-\cdot950$, we find

$$x=\cdot840\pm\cdot494\sqrt{-1}=\alpha\pm\beta\sqrt{-1}$$

where $(\alpha^2+\beta^2)=\cdot950$ and $\beta/\alpha=\tan^{-1}\ 30°\cdot44$ leading to the solution $u_t=A\,(\cdot950)^{t/2}\cos(30°\cdot44\,t+a)$ where t is measured in tenths of a year.

The period is $360°/304°\cdot4$ years $=$ 432 days and the amplitude will diminish to half its value in 32 months.

The years 1912 to 1927 were treated in a similar manner with the slight difference that the means for each tenth of a year for the 14 years 1912—1925 were taken out instead of a simple harmonic term, giving the result

$$u_t=A\,(\cdot855)^{t/2}\cos(31°\cdot3t+a)$$

corresponding to a period of 420 days and a very rapid modulus of decay.

But as the years 1926 and 1927 are, as seen from the diagram, very different from previous years, a fresh solution was made using only the years 1912—1925 from which the extracted annual period had been obtained.

This gave $u_t=A\,(\cdot936)^{t/2}\cos(30°\cdot8\,t+a)$ corresponding to a period of 427 days.

3. As the rate of decay indicated by these solutions was much too rapid, fresh solutions were made on the assumption that $q=1$; *i.e.*, that the vibration was of simple harmonic character. The angular movements in a tenth of a year were found to be $30°\cdot7$, $31°\cdot7$ and $30°\cdot9$ in the three cases, corresponding to periods of 428, 415 and 424 days.

4. Next the two sets of quantities $u_{t+2}+u_t-2\cos30°\cdot u_{t+1}$ and $u_{t+2}+u_t-2\cos31°\cdot u_{t+1}$ were formed for each tenth of a year from the beginning to the end of the series.

These correspond to the formation of $\dfrac{d^2x}{dt^2}+k^2x$ for the two cases where the free period corresponds to a movement of 30° or 31° in a tenth of a year; *i.e.*, to periods of 438 and 424 days.

These were worked out separately for the years 1900—1911 and 1912—1927. In order to show the size of the quantities and their irregularities the figures are given for the period 1900—1911 for $u_{t+2}+u_t-2\cos 30^\circ\cdot u_{t+1}$ in Table 2.

TABLE II.

$u_t+u_{t+2}-2\cos 30^\circ\cdot u_{t+1}$ (Unit $''\cdot 001$).

	·05	·15	·25	·35	·45	·55	·65	·75	·85	·95
1900	—	−17	+23	−13	+ 9	−13	− 6	−26	+41	−35
01	+20	+ 1	+29	+ 9	−15	+17	+ 6	−17	+24	− 4
02	+23	− 6	+47	+ 6	+23	−34	+ 6	+14	−10	+28
03	0	+27	− 7	+15	− 9	+30	−24	+ 1	+ 2	−15
04	+38	+ 7	+14	− 2	− 1	− 7	+12	−24	+30	−38
05	+ 5	+39	−12	+22	− 7	+19	−27	− 8	+32	−72
06	+43	+17	−23	+13	− 9	+ 3	−16	− 5	−15	+11
07	−14	+33	− 4	−17	+23	−23	−43	+28	+25	−21
08	+42	−11	+38	− 4	+ 9	− 2	−16	+ 9	−20	−14
09	+45	−14	+29	+ 4	− 6	−14	0	−26	+35	+35
10	−46	− 5	+45	+17	− 5	−19	− 2	+ 3	− 3	+ 2
11	+19	+14	+56	−51	+52	−20	+ 2	+16	+12	—
Mean	+15	+7	+19	0	+ 5	− 5	− 9	− 2	+13	−11

Harmonic analysis of these means gives

$$+''\cdot 0086\cos(36^\circ t-55^\circ)-''\cdot 0034\cos(72^\circ t-80^\circ).$$

Assuming that

$$u_t=A\cos(30^\circ t+\alpha)+B\cos(36^\circ t+\beta)+C\cos(72^\circ t+\gamma)$$

we find that $u_{t+2}+u_t-2\cos 30^\circ\cdot u_{t+1}$

$$=2B(\cos 36^\circ-\cos 30^\circ)\cos(36^\circ t+\beta)+2C(\cos 72^\circ-\cos 30^\circ)\cos(72^\circ t+\gamma).$$

$\cos 36° - \cos 30° = -\cdot 057 + \cos 72° - \cos 30° = -\cdot 557.$

Equating to the numerical values found above

$$B = -''\cdot 076 \text{ and } C = +''\cdot 003.$$

The effect of resonance is shown in the increase of the annual term. It magnifies this term in the ratio

$$\frac{36^2}{36^2 - 30^2} \text{ or about 3 times.}$$

5. Whether a movement of 30° or 31° in a tenth of a year is assumed for the free period, the annual period determined in the above manner is practically the same. For the period 1900—1911 we find

$$-''\cdot 076 \cos (36°t - 55°) + ''\cdot 003 \cos (72°t - 80°)$$

$$\text{or} \quad -''\cdot 070 \cos (36°t - 55°) + \cdot 004 \cos (72°t - 84°)$$

while for the period 1912—1927 the results are

$$-''\cdot 105 \cos (36°t - 73°) + ''\cdot 017 \cos (72°t - 97°)$$

$$\text{and } -''\cdot 106 \cos (36°t - 73°) + ''\cdot 017 \cos (72°t - 96°)$$

on the two assumptions of the length of the free period.

The value of the annual term as derived directly from the means of the values of u_t for the period 1912—1927 is

$$-''\cdot 118 \cos (36°t - 68°) + ''\cdot 016 \cos (72°t - 86°).$$

6. The quantities in Table 2 are necessarily subject to an accidental error which is large compared with the small unit $''\cdot 001$. The quantity $u_{t+2} + u_t - 2 \cos 30° \cdot u_{t+1}$ being practically a second difference is necessarily liable to be somewhat irregular. As indicating this it may be noticed that the largest quantity in Table 2 ($-''\cdot 072$ for 1905·95) is preceded by $+''\cdot 032$ and followed by $+''\cdot 043$. The figures in this table contain the actual disturbing force but in individual cases it is obliterated by observational errors.

7. In Table 3 is given the mean of the quantities in Table 1 for successive periods of 7 years.

TABLE III.

Means for successive periods of 7 years.

	·05	·15	·25	·35	·45	·55	·65	·75	·85	·95
00–06	−81	− 54	− 53	− 27	+11	+ 46	+ 70	+ 69	+40	+16
01–07	−31	− 54	− 46	− 19	+19	+ 51	+ 71	+ 60	+31	+ 7
02–08	−86	− 50	− 36	− 4	+31	+ 63	+ 76	+ 53	+19	−14
03–09	−61	− 70	− 46	− 4	+43	+ 79	+ 94	+ 69	+21	−19
04–10	−70	− 87	− 71	− 24	+34	+ 84	+106	+ 86	+40	− 4
05–11	−61	− 89	− 81	− 34	+20	+ 77	+106	+ 88	+53	+13
06–12	−45	− 77	− 87	− 49	− 1	+ 54	+ 78	+ 62	+32	− 1
07–13	−38	− 52	− 56	− 19	+23	+ 69	+ 80	+ 49	+ 6	−34
08–14	−64	− 65	− 61	− 18	+41	+ 90	+ 99	+ 69	+16	−43
09–15	−91	−109	−105	− 51	+22	+100	+132	+100	+43	−22
10–16	−81	−118	−130	− 88	−15	+ 73	+121	+107	+69	+ 5
11–17	−68	−111	−126	− 97	−33	+ 52	+ 99	+ 89	+60	+ 9
12–18	−53	− 90	−109	− 96	−32	+ 48	+ 82	+ 70	+37	−10
13–19	−56	− 86	− 96	− 74	− 2	+ 77	+109	+ 81	+38	−25
14–20	−78	−113	−117	− 79	+ 9	+103	+140	+112	+52	−16
15–21	−81	−125	−132	− 92	+ 5	+107	+152	+126	+57	−15
16–22	−73	−115	−129	− 96	− 4	+ 96	+147	+126	+56	−11
17–23	−65	−107	−127	−107	−25	+ 81	+136	+113	+38	−22
18–24	−60	− 95	−117	−102	−32	+ 61	+115	+ 90	+16	−39
19–25	−70	− 95	−112	− 89	−22	+ 57	+119	+ 88	+ 4	−59
20–26	−92	−107	−109	− 82	−32	+ 61	+115	+ 94	+21	−40
21–27	−75	− 96	−108	− 91	−39	+ 41	+101	+ 87	+28	−27

These figures have been analysed harmonically but only those results for the periods 1900—1908, 1907—1913, 1914—1920, 1921—1927 are given here.

1900–06	$+''{\cdot}009$	$-''{\cdot}064 \cos (36^\circ t - 71^\circ)$	$+''{\cdot}002 \cos (72^\circ t + 24^\circ)$
1907–13	$+{\cdot}007$	$-{\cdot}067 \cos (36^\circ t - 49^\circ)$	$+{\cdot}011 \cos (72^\circ t - 73^\circ)$
1914–20	$+{\cdot}001$	$-{\cdot}128 \cos (36^\circ t - 66^\circ)$	$+{\cdot}014 \cos (72^\circ t - 74^\circ)$
1921–27	$-{\cdot}018$	$-{\cdot}102 \cos (36^\circ t - 74^\circ)$	$+{\cdot}017 \cos (72^\circ t - 120^\circ)$

The annual term has increased considerably in amount and apparently reached a maximum value about 1918. The range in semi-amplitude of the principal term is from $''\cdot060$ to $''\cdot142$ and in phase from 45° to 78°.

8. When the quantities in Table 3 are subtracted from those in Table 1, the annual period is taken out as far as possible. Eor the years 1900,01,02,03, the means for the years 1900—1906 are taken, for 1904 the means for 1901—1907 are taken and so on. The three years at the beginning and end are necessarily not treated so satisfactorily as the intermediate years. The free periods are thus obtained in Table 4 and in diagram 2.

TABLE IV.

Free Period of the Variation of Latitude.

	·05	·15	·25	·35	·45	·55	·65	·75	·85	·95
1900	+ 91	+104	+ 63	+ 17	− 51	− 96	−130	−129	−110	− 86
1	+ 31	+ 94	+123	+137	+119	+ 54	− 10	− 59	−100	−106
2	− 69	− 6	+ 43	+117	+149	+164	+100	+ 21	− 40	−116
3	−119	−106	− 47	+ 7	+ 69	+104	+140	+121	+ 80	+ 4
4	− 69	−106	−124	−101	− 59	− 01	+ 49	+110	+119	+113
5	+ 56	− 30	− 84	−136	−131	−103	− 26	+ 47	+101	+154
6	+111	+ 60	− 4	− 96	−153	−179	−154	− 89	− 1	+ 59
7	+130	+137	+131	+ 74	− 24	− 94	−156	−206	−170	− 76
8	+ 31	+159	+221	+244	+200	+103	− 16	−128	−203	−253
9	−235	−123	+ 7	+139	+241	+266	+222	+138	− 12	−129
10	−172	−228	−224	−141	− 3	+121	+210	+261	+244	+154
11	+ 24	−105	−179	−177	−181	− 90	+ 21	+114	+222	+283
12	+223	+113	− 51	−191	−268	−300	−274	−180	− 65	+ 62
13	+163	+278	+202	+198	+ 75	− 68	−171	−215	−235	−190
14	− 56	+ 69	+154	+187	+160	+ 82	− 12	− 71	−120	−153
15	−163	−143	− 61	+ 41	+119	+207	+238	+108	+ 1	− 88
16	−152	−180	−159	− 98	− 18	− 54	+118	+169	+171	+ 85
17	− 44	−115	−137	−142	−117	− 58	− 1	+ 71	+133	+163
18	+146	+ 98	+ 16	− 88	−141	−169	−153	− 78	+ 23	+123
19	+184	+147	+ 61	+ 6	− 29	− 57	− 99	−125	−109	− 55

TABLE IV—*contd*

	·05	·15	·25	·35	·45	·55	·65	·75	·85	·95
20	+ 14	+ 83	+146	+182	+161	+103	+ 30	− 8	− 68	− 98
21	− 85	− 33	+ 34	+100	+188	+102	+ 55	+28	− 39	− 94
22	− 94	− 71	− 33	+ 4	+ 45	+108	+169	+92	+ 26	− 11
23	− 57	− 99	−134	−165	−139	− 37	+ 29	+59	+ 54	+ 22
24	− 14	− 47	− 73	− 97	−119	−137	−114	−60	+ 8	+ 50
25	+ 74	+ 65	+ 31	+ 7	− 30	− 60	− 73	−57	− 34	− 3
26	+ 32	+ 47	+ 58	+ 17	+ 8	− 43	− 76	−39	+ 34	+ 96
27	+144	+149	+135	+105	+ 56	+ 2	− 34	−31	− 6	0

9. Inspection of diagram 2 shows that the amplitude of the free vibration was considerably greater from 1907 to 1918 than in the years 1900 to 1907. The period is very regular till 1920. There is then a short period in 1921 followed by a long and irregular one 1922—23, and from then till the end of 1927 a large drop in the amplitude. Examination of diagram 2 shows a very small and rather early maximum at 1923·75. If the quantities $u_t + u_{t+2} - 2u_{t+1} + 2(1-\cos 30°)u_{t+1}$ be formed, it will be found that they are $-''·058$, $-''·073$ and $-''·046$ for the times 1923·55, 1923·65, 1923·75 This agrees with the supposition that in the autumn of 1923 a force acted which prevented the oscillation from reaching its normal maximum and thus the amplitude of the free period was diminished.

10. In the *Monthly Notices* for May, 1898, the length of the free period was determined from observations from 1841 to 1911 to be 432 days. The present series would give 428 days, while the 73 periods from 1841·46 to 1927·85 give 1·180 years or 431 days Carrying on the period given in *M.N.* we have for the observed and calculated dates of minimum:

Obs.	Cal.	O−C.	Obs.	Cal.	O−C.
1903·04	·08	−·04	1917·30	·28	+·02
06·52	·63	−·11	20·96	·83	+·13
10·18	·18	·00	24·55	·38	+·15
13·83	·73	+·10	27·70	·93	−·23

The minimum at 1927·70 is not well determined.

In conclusion I should like to express my indebtedness to Mr. Acton for his help in the numerical work and the careful diagrams he has prepared.

SULLE EQUAZIONI FUNZIONALI DEL TIPO DI VOLTERRA

DI

LEONIDA TONELLI (*Bologna*).

[*Communicated by Professor Ganesh Prasad, 29th July, 1928.*]

Consideriamo il *funzionale*

$$(1) \qquad A[x, \phi_{y_1}^{y_2}(y)]:$$

esso rappresenta ***un numero reale***, dipendente, secondo una data legge, esclusivamente dalla x considerata e dai valori assunti dalla funzione $\phi(y)$ nell'intervallo (y_1, y_2) (ove supponiamo $y_1 \leq y_2$). Intenderemo che il funzionale (1) sia definito per ogni x (reale) dell'intervallo $(0, 1)$; per ogni y_1 (reale) pure di $(0, 1)$; per ogni y_2 (reale) dell'intervallo $(y_1, 1)$; per ogni funzione (reale) $\phi(y)$, data nell' intervallo (y_1, y_2) e in esso continua e sempre soddisfacente alle disuguaglianze

$$a \leq \phi(y) \leq b.$$

Sottoponendo il funzionale (1) ad opportune ipotesi, che preciseremo nel seguito, studieremo *l'equazione funzionale del tipo di Volterra*

$$(E) \qquad \phi(x) = f(x) + A[x, \phi_0^x(y)]$$

—dove $f(x)$ è una funzione data nell'intervallo $(0, 1)$—proponendoci di dimostrare l'esistenza della sua soluzione e di stabilire poi alcune proprietà fondamentali di tale soluzione Mostreremo, infine, che il metodo usato da Cauchy, per provare l'esistenza dell'integrale di un'equazione differenziale ordinaria, può servire anche per l'equazione funzionale (E).

Come casi particolari dell' equazione (E), indicheremo le equazioni integrali di Volterra

$$(E_1) \qquad \phi(x) = f(x) + \int_0^x K\Big(x, y, \phi(y)\Big)\, dy,$$

$$(E_2) \qquad \phi(x)=f(x)+\int_0^x \mathrm{K}(x,\, y)\phi(y)dy,$$

$$(E_3) \qquad \phi(x)=c+\int_0^x \mathrm{K}\Big(y,\ \phi(y)\Big)\,dy,$$

all'ultima delle quali si riconduce l'equazione differenziale ordinaria

$$y'=\mathrm{K}(x,\, y).$$

Altri casi particolari sono dati dalle equazioni

$$(E_4) \qquad \phi(x)=f(x)+\int_0^x\int_0^x \mathrm{K}\,(x,\, y,\, z)\phi(y)\phi(z)dydz,$$

$$(E_5) \qquad \phi(x)=f(x)+\int_0^x \mathrm{K}\Big(x,\, y,\ \phi(y)\Big)\,dy$$

$$+\int_0^x\int_0^x \mathrm{H}\Big(x,\, y,\, z,\ \phi(y),\, \phi(z)\Big)\,dydz.$$

1. Supporremo sempre, in ciò che segue, salvo dirrso in contrario, che il funzionale (1) soddisfialle tre condizioni che ora indicheremo.

(I) Esiste un numero M_1 tale che, per ogni x di $(0, 1)$, per ogni intervallo (y_1, y_2) di $(0, 1)$, e per ogni funzione $\phi(y)$, continua in (y_1, y_2) ed ivi soddisfacente alle disuguaglianze $a \leq \phi(y) \leq b$, sia

$$(C_1) \qquad |\ \mathrm{A}[x,\, \phi_{y_1}^{y_2}(y)]\ | \leq \mathrm{M}_1(y_2-y_1).$$

(II) Esiste un numero M_2, tale che, per ogni x di $(0, 1)$, per ogni terna y_1, y_2, y_3, con $o \leq y_1 \leq y_2 \leq y_3 \leq 1$, e per ogni funzione $\phi(y)$, continua in (y_1, y_3) ed ivi soddisfacente alle $a \leq \phi(y) \leq b$, sia

$$(C_2) \qquad | A[x, \phi_{y_1}^{y_3}(y)] - A[x, \phi_{y_1}^{y_2}(y)] | \leq M_2(y_3 - y_2).$$

(III) Ad ogni $\epsilon > 0$, si può sempre far corrispondere un $\rho > 0$, in modo che, se è $o \leq x_1 \leq x_2 \leq 1$, $x_2 - x_1 \leq \rho$, e se $\phi_1(y)$ e $\phi_2(y)$ sono due funzioni, definite e continue in un qualsiasi intervallo (y_1, y_2) di $(0, 1)$, ambedue sempre comprese fra a e b, e tali che in (y_1, y_2), sia sempre $| \phi_1(y) - \phi_2(y) | \leq \rho$, risulti

$$(C_3) \qquad | A[x_1, \phi_1{}_{y_1}^{y_2}(y)] - A[x_2, \phi_2{}_{y_1}^{y_2}(y)] | \leq \epsilon\, (y_2 - y_1).$$

Supporremo sempre, inoltre, che la $f(x)$ sia una funzione data e continua in tutto l'intervallo $(0, 1)$, e soddisfacente alla condizione

$$a < f(o) < b.$$

2. Vogliamo dimostrare che:

per un certo valore $l > 0$, *l'equazione* (E) *ammette almeno una soluzione (continua).*

A tal fine, considerato un qualunque numero intero positivo n, definiamo la funzione $\phi_n(x)$, ponendo

$$(2) \qquad \begin{cases} \phi_n(x) = f(x), & \text{per } o \leq x \leq \dfrac{1}{n}, \\ \phi_n(x) = f(x) + A[x, \phi_n{}_0^{x - \frac{1}{n}}(y)], & \text{per } \dfrac{1}{n} < x. \end{cases}$$

Questa funzione risulta così definita in un intervallo $(0, x_0)$, x_0 essendo il massimo numero di $(0, 1)$ tale che, per ogni x di $\left(0, x_0 - \frac{1}{n}\right)$, sia

$$(3) \qquad a \leq \phi_n(x) \leq b.$$

Mostriamo che, qualunque sia n, x_0 resta maggiore di un numero fisso, non nullo. Detto 2δ il minore dei due numeri $f(o)-a$ e $b-f(o)$, M il minore dei numeri M_1 e M_2, indichiamo con l il massimo numero positivo non superiore nè ad 1 nè a $\delta:M$, e tale che, in tutto l' intervallo $(0, l)$, risulti

$$|f(x) - f(o)| \leq \delta.$$

Indichiamo poi con r l' intero tale che $\frac{r}{n} \leq l < \frac{r+1}{n}$. Allora, dalle (2) e tenendo conto di (C_1), segue :

$$\text{per } 0 \leq x \leq \frac{1}{n}, \qquad a \leq \phi_n(x) \leq b,$$

$$\text{,,} \quad \frac{1}{n} < x \leq \frac{2}{n}, \qquad \phi_n(x) \geq f(o) - \delta - Ml \geq a.$$

$$\phi_n(x) \leq f(o) + \delta + Ml \leq b,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\text{,,} \quad \frac{r}{n} < x \leq \frac{r+1}{n}, \qquad \phi_n(x) \geq f(o) - \delta - Ml \geq a,$$

$$\phi_n(x) \leq f(o) + \delta + Ml \leq b.$$

È dunque $x_0 \geq \frac{r+1}{n} > l$. Perciò la funzione $\phi_n(x)$ risulta definita in tutto l' intervallo $(0, l)$ ed ivi soddisfa sempre alle disuguaglianze (3). Inoltre,

preso ad arbitrio un $\epsilon > 0$, e determinato un ρ_1 positivo, minore del ρ indicato nella condizionene (III) del $n^\circ 1$, minore di $\epsilon : M$, e tale che dalle $o \leq x' < x'' \leq 1$, $x'' - x' \leq \rho_1$, segua $| f(x') - f(x'') | < \epsilon$. abbiamo, dalle (2) e da (C_2) e (C_3), purchè sia $o \leq x' < x'' \leq l$ e $x'' - x' \leq \rho_1$,

$$| \phi_n(x') - \phi_n(x'') | < | f(x') - f(x'') |$$

$$+ | A[x', \phi_n{}_0^{x'-\frac{1}{n}}(y)] - A[x'', \phi_n{}_0^{x'-\frac{1}{n}}(y)] |$$

$$+ | A[x'', \phi_n{}_0^{x'-\frac{1}{n}}(y)] - A[x'', \phi_n{}_0^{x''-\frac{1}{n}}(y)] |$$

$$< \epsilon + \epsilon \left(x' - \frac{1}{n} \right) + M(x'' - x') < 3\,\epsilon.$$

Risulta così provato che le funzioni $\phi_n(x)$ $(n=1, 2,...)$ sono, nell'intervallo $(0, l)$, tutte ugualmente limitate ed ugualmente continue Esse ammettono, perciò, in $(0, l)$, almeno una funzione limite continua $\phi_\infty(x)$: esiste cioè almeno una successione $\phi_{n_1}(x)$, $\phi_{n_2}(x)$, ..., $\phi_{n_m}(x)$, ..., estratta da quella delle $\phi_n(x)$, uniformemente convergente, in tutto $(0, l)$, verso una funzione continua $\phi_\infty(x)$.

Affermiamo che la $\phi_\infty(x)$ è, in $(0, l)$, una soluzione dell'equazione (E). La $\phi_\infty(x)$ è, infatti, come limite di funzioni tutte comprese fra a e b, anche essa sempre compresa fra tali limiti. Inoltre, preso ad arbitrio un $\epsilon > 0$, e considerato il numero ρ ad esso corrispondente secondo la condizione (III) del $n^\circ 1$, possiamo scegliere un $\overline{m}$ tale che, per ogni $m > \overline{m}$ sia, in $(0, l)$,

$$| \phi_{n_m}(x) - \phi_\infty(x) | < \rho.$$

Possiamo anche supporre $\overline{m}$ tale che, per $m > \overline{m}$, sia $1 : n_m < \epsilon : M$. Poichè, per $x=0$ è $\phi_n(o) = f(o)$ e quindi $\phi_\infty(o) = f(o)$, la $\phi_\infty(x)$ verifica la (E) per $x=0$. Se è $o < x \leq l$, e supposto $\overline{m}$ sufficientemente grande affinchè,

per $m > \bar{m}$, sia pure $1 : n_m < x$, abbiamo, per ogni $m > \bar{m}$, in virtù di (C_2) e (C_3),

$$| A[x, \phi_{n_{m_0}}^{x-\frac{1}{n_m}}(y)] - A[x, \phi_{\infty_0}^{x}(y)] |$$

$$\leq \epsilon \left(x - \frac{1}{n_m} \right) + M\frac{1}{n_m} < 2\epsilon,$$

e perciò

$$| \phi_\infty(x) - f(x) - A[x, \phi_{\infty_0}^{x}(y)] |$$

$$< | \phi_\infty(x) - \phi_{n_m}(x) | + | \phi_{n_m}(x) - f(x) - A[x, \phi_{n_{m_0}}^{x-\frac{1}{n_m}}] |$$

$$+ | A[x, \phi_{n_{m_0}}^{x-\frac{1}{n_m}}(y)] - A[x, \phi_{\infty_0}^{x}(y)] | < \rho + 2\epsilon,$$

e poichè possiamo ritenere ρ minore di ϵ, ne viene, per l'arbitrarietà di ϵ,

$$\phi_\infty(x) - f(x) - A[x, \phi_{\infty_0}^{x}(y)] = 0,$$

il che prova precisamente che la $\phi_\infty(x)$ è soluzione della (E) in tutto $(0, l)$.

3. Le ipotesi generali del n°1 non sono sufficienti ad assicurare l'unicità della soluzione dell' equazione (E), Ed infatti, se supponiamo

$$f(x) \equiv 0,$$

$$A[x, \phi_{y_1}^{y_2}(y)] \equiv \int_{y_1}^{y_2} \sqrt[3]{\phi(y)}\, dy,$$

l'equazione (E) diventa

$$\phi(x)=\int_0^x \sqrt[3]{\phi(y)}\, dy,$$

e questa equazione ammette, in un intorno a destra del punto $x=0$, la soluzione $\phi(x)\equiv 0$ ed anche le infinite altre soluzioni date da

$$\phi(x)=0, \quad \text{in } (0, c),$$

$$\phi(x)= \pm\left[\frac{2}{3}(x-c)\right]^{\frac{3}{2}}, \quad \text{per } x > c,$$

con c costante positiva arbitraria.*

4. Supponiamo che il funzionale (1) soddisfi, oltre che alle condizioni poste nel n^o 1, anche alla nuova condizione seguente:

(IV) Esiste un numero M' tale che, per ogni x di $(0, 1)$, per ogni terna y_1, y_2, y_3, con $o \leq y_1 \leq y_2 \leq y_3 \leq 1$, e per ogni coppia di funzioni $\phi_1(y)$ e $\phi_2(y)$, continue in (y_1, y_3) ed ivi sempre comprese fra a e b, sia

$$(C_4) \qquad \left| A\left[x, \phi_1{}_{y_1}^{y_3}(y)\right] - A\left[x, \phi_2{}_{y_1}^{y_3}(y)\right]\right|$$

$$\leq M' \left\{(y_2-y_1)\max\left|\phi_1{}_{y_1}^{y_3}-\phi_2\right|\right.$$

$$\left. +(y_3-y_1)\max\left|\phi_1{}_{y_2}^{y_3}-\phi_2\right|\right\},$$

dove $\max \mid \phi_1{}_{y'}^{y''}-\phi_2 \mid$ rappresenta il massimo valore di $\mid \phi_1(y)-\phi_2(y) \mid$ nell'intervollo (y', y'').

* Cfr. P. Montel—*Sur les suites infinies de fonctions.* (Ann. École Norm. Sup., 1907.)

Possiamo, allora, dimostrare che *la soluzione dell' equazione* (E) *è necessariamente unica.*

Ammettiamo, infatti, che, in un intervallo $(0, \lambda)$ di $(0, 1)$, esistano due soluzioni distinte, $\phi^{(1)}(x)$ e $\phi^{(2)}(x)$,* della (E). Osservando che dalla (C_1) segue

$$A[x, \phi_0^0(y)]=0,$$

abbiamo

$$\phi^{(1)}(0)=f(o),$$

$$\phi^{(2)}(0)=f(o).$$

Dopo di ciò, preso ad arbitrio un $\epsilon > 0$, indichiamo con x' la massima ascissa contenuta in $(0, \lambda)$, tale che, in tutto $(0, x')$, sia

$$|\phi^{(1)}(x)-\phi^{(2)}(x)| \leq \epsilon.$$

Sarà $x'=\lambda$ oppure $|\phi^{(1)}(x')-\phi^{(2)}(x')|=\epsilon$, e, in questo secondo caso, avendosi, per la (E),

$$(4)\quad \phi^{(1)}(x')-\phi^{(2)}(x')=A\left[x', \phi_0^{(1)^{x'}}(y)\right]-A\left[x', \phi_0^{(2)^{x'}}(y)\right],$$

dovrà essere, per (C_4),

$$\epsilon \leq M' x' \epsilon,$$

e quindi

$$x' \geq \frac{1}{M'}.$$

* Siccome il funzionale (1) è stato definito soltanto per le funzioni $\phi(y)$ continue, intendiamo che $\phi^{(1)}(x)$ e $\phi^{(2)}(x)$ siano funzioni continue.

E sicome ϵ è arbitrario, dovrà aversi, in ogni caso, in tutto l'intervallo $\left(0, \frac{1}{M'}\right)$, $\phi^{(1)}(x) \equiv \phi^{(2)}(x)$. Se, perciò, non è $x'=\lambda$, dovrà aversi, ancora per la (4) e in forza di (C_4),

$$\epsilon \leq M'\left(x' - \frac{1}{M'}\right)\epsilon,$$

donde

$$x' \geq \frac{2}{M'},$$

e, in ogni caso, sempre per l'arbitrarietà di ϵ, dovrà essere $\phi^{(1)}(x) \equiv \phi^{(2)}(x)$ tutto l'intervallo $\left(0, \frac{2}{M'}\right)$. Così proseguendo, si vede che l'identità $\phi^{(1)}(x) \equiv \phi^{(2)}(x)$ deve sussistere in tutto $(0, \lambda)$.

5. Nelle condizioni del n° preced., *la successione* $\phi_1(x)$, $\phi_2(x)$,......, *costruita nel* n° 2, *converge uniformemente, nell'intervallo* $(0, l)$, *determinato nel* n° *detto, verso la* $\phi_\infty(x)$.

Supponiamo che ciò non sia vero. Allora deve esistere almeno un $\sigma > 0$ tale che, per ogni n, uno almeno dei valori di n maggiori di $\bar{n}$ renda soddisfatta, in qualche punto di $(0, l)$, lo disuguaglianza

$$|\phi_n(x) - \phi_\infty(x)| > \sigma;$$

devono perciò esistere una successione di numeri interi crescenti $n_1, n_2, \ldots\ldots, n_r, \cdots$ ed una successione di punti $x_1, x_2, \ldots x_r, \ldots$ di $(0, l)$, in modo da aversi

$$(5) \qquad |\phi_{n_r}(x_r) - \phi_\infty(x_r)| > \sigma.$$

Siccome le $\phi_n(x)$ sono tutte ugualmente continue ed ugualmente limitate in $(0, l)$. dalla successione $\phi_{n_r}(x)$, $(r=1, 2, \ldots)$, ne possiamo estrarre un'altra

$\phi_{n'}(x)$, $\phi_{n''}(x)$,... convergente uniformemente in tutto $(0, l)$ verso un
one continua $\overline{\phi}_\infty(x)$; e per la (5) possiamo affermare che, in alr
punto di $(0, l)$, deve essere

$$|\overline{\phi}_\infty(x) - \phi_\infty(x)| \geq \sigma.$$

Ma la $\overline{\phi}_\infty(x)$ è, come la $\phi_\infty(x)$, una soluzione dell'equazione
$(0, l)$: esisterebbero così due soluzioni distinte della (E), contro quant
dimostrato nel $n^\circ 4$.

6. Sotto le ipotesi del $n^\circ 1$ e quella del $n^\circ 4$, possiomo mostr
se esiste, in un intervallo $(0, \lambda)$ *di* $(0, 1)$, *una soluzione* $\phi^*(x)$ *della* (E),
soddisfacente alle disuguaglianze $a < \phi^*(x) < b$, *la funzione* $\phi_n(x)$,
nel n°2, *converge uniformemente, per* $n \longrightarrow \infty$, *verso la* $\phi^*(x)$ in tutto ((

Per la dimostrazione di questa proposizione, cominciamo con l'os
che, indicato con 4Δ il minore dei minimi di $b-\phi^*(x)$ e $\phi^*(x)-a$ in
dalla convergenza uniforme di $\phi_n(x)$ verso $\phi^*(x)$, in tutto $(0, l)$, conv
già stabilita nel $n^\circ 5$, segue l'esistenza di un numero n_1, tale che, p
$n > n_1$, sia, in $(0, l)$,

$$Q + 3\Delta < \phi_n(x) < b - 3\Delta. \tag{6}$$

Determiniamo, poi, un numero l_1, maggiore di zero, minore di Δ
tale che se è $o \leq x' < x'' \leq 1$, $x''-x' \leq l_1$, sia

$$|f(x') - f(x'')| < \Delta,$$

$$\left| A\left[x', \phi_{y_1}^{y_2}(y)\right] - A\left[x'', \phi_{y_1}^{y_2}(y)\right] \right| < \Delta(y_2 - y_1),$$

per qualsiasi funzione $\phi(y)$ definita in uno qualsiasi degli intervalli (
di $(0, 1)$, e ivi sempre continua e compresa fra a e b. La possibilità
determinazione di l_1 risulta dalla continuità della $f(x)$ e da (C_3).

Essendo ancora $\frac{r}{n} \leq l < \frac{r+1}{n}$, sia $\frac{s}{n} \leq l_1 < \frac{s+1}{n}$. Allora, supposto $n > n_1$ è

$$\phi_n(x) = \left\{ f(l) + \mathrm{A}\left[l, \phi_{n}{}_0^{l-\frac{1}{n}}(y) \right] \right\} + \left\{ f(x) - f(l) \right\}$$

$$+ \left\{ \mathrm{A}\left[x, \phi_{n}{}_0^{l-\frac{1}{n}}(y) \right] - \mathrm{A}\left[l, \phi_{n}{}_0^{l-\frac{1}{n}}(y) \right] \right\}$$

$$+ \left\{ \mathrm{A}\left[x, \phi_{n}{}_0^{x-\frac{1}{n}}(y) \right] - \mathrm{A}\left[x, \phi_{n}{}_0^{l-\frac{1}{n}}(y) \right] \right\},$$

e per $l \leq x \leq \frac{r+1}{n}$, in forza delle disuguaglianze sopra scritte e di (C_2),

$$\phi_n(x) < \{b - 3\Delta\} + \Delta + \Delta\left(l - \frac{1}{n}\right) + \mathrm{M}l_1 < b,$$

ed analogamente

$$\phi_n(x) > Q.$$

Per $\frac{r+1}{n} < x \leq \frac{r+2}{2}$, avremo ancora

$$a < \phi_n(x) < b,$$

e queste disuguaglianze varranno per ogni $x \leq l + l_1$. Ne segue che le funzioni $\phi_n(x)$ sono, per $n > n_1$, tutte ugualmente limitate e quindi anche (per un ragionamento fatto nel n°2) ugualmente continue, in tutto l'intervallo $(0, l + l_1)$: esse, dunque, convergono, in tale intervallo, per $n \longrightarrow \infty$, verso l'unica soluzione $\phi^*(x)$ della (E). E' pertanto possibile di determinare un numero $n_2 > n_1$, tale che, per ogni $n > n_2$, valga la (6) in tutto l'intervallo $(0, l + l_1)$. Ripetendo il ragionamento fatto or ora, si vede che la (6) vale anche, per tutti gli n maggiori di un certo $n_3 > n_2$, e per ogni x di

$(0, l+l_1+l_2)$. Così proseguendo, si ottiene che la (6) vale, per tutti gli n maggiori di un certo $\bar{n}$ nell'intero intervallo $(0, \lambda)$ in cui esiste la soluzione $\phi^*(x)$.

7. Supponiamo che sia $a=-\infty$ e $b=+\infty$, che valgano la condizione (IV) (n° 4) e la $a < f(o) < b$, e che, invece delle (I), (II), (III), vagano le seguenti condizioni:

(I') Ad ogni numero intero positivo m, corrisponde un numero $M_{1,m}$, tale che, per ogni x di $(0, 1)$, per ogni intervallo (y_1, y_2) di $(0, 1)$, e per ogni funzione $\phi(y)$, continua in (y_1, y_2) ed ivi soddisfacente alla $|\phi(y)| \leq m$, sia

$$(C'_1) \qquad \left| A\left[x, \phi_{y_1}^{y_2}(y)\right]\right| \leq M_{1,m}\,(y_2-y_1).$$

(II') Ad ogni intero positivo m, corrisponde un $M_{2,m}$, tale che, per ogni terna y_1, y_2, y_3, con $0 \leq y_1 \leq y_2 \leq y_3 \leq 1$, e per ogni $\phi(y)$, continua in (y_1, y_3) ed ivi soddisfacente alla $|\phi(y) \leq m$, sia

$$(C'_2) \qquad \left| A\left[x, \phi_{y_1}^{y_3}(y)\right] - A\left[x, \phi_{y_1}^{y_2}(y)\right]\right| \leq M_{2,m}\,(y_3-y_2).$$

(III') Ad ogni intero positivo m, e ad ogni $\epsilon > 0$, corrisponde un $\rho_m > 0$, in modo che, se è $0 \leq x_1 \leq x_2 \leq 1$, $x_2-x_1 \leq \rho_m$, e se $\phi_1(y)$ e $\phi_2(y)$ sono due funzioni continue in un intervallo qualsiasi (y_1, y_2) di $(0, 1)$, ambedue sempre, in modulo, minori od uguali ad m, e tali che, in (y_1, y_2), sia sempre $|\phi_1(y)-\phi_2(y)| \leq \rho_m$, risulti verificata la (C_3).

Con queste ipotesi, *l'unica soluzione* $\phi^*(x)$ *della equazione* (E) *esiste* (*sempre continua*) *in tutto l'intervallo* $(0, 1)$.

Indichiamo, infatti, con $(0, \lambda)$ il massimo intervallo di $(0, 1)$ nell'*interno* del quale la $\phi^*(x)$ esiste finita e continua, con N il massimo modulo della $f(x)$ in $(0, 1)$, con Φ quello della $\phi^*(x)$ in $\left(0, \lambda-\frac{1}{2M'}\right)$, e con $\Psi(\lambda)$ quello della $\phi^*(x)$ in $\left(\lambda-\frac{1}{2M'}, x\right)$. Abbiamo, con ciò, dalla (E) e da (C_4),

per $\lambda-\frac{1}{2M'} \leq x < \lambda$ (e intendendo di sostituire a $\lambda-\frac{1}{2M'}$ lo 0, se è $2M'\lambda < 1$)

$$\left| \phi^*(x) \right| \leq \left| f(x) \right| + \left| A\left[x,\ 0_0^x\right] \right|$$

$$+M'\left\{\left(\lambda-\frac{1}{2M'}\right)\Phi+\frac{1}{2M'}\ \Psi(x)\right\},$$

e quindi, tenendo conto di (C'_1),

$$|\ \phi^*(x)\ | \leq N+M_{1,\ 0}+M'\Phi+\tfrac{1}{2}\Psi(x),$$

donde

$$\Psi(x) \leq N+M_{1,\ 0}+M'\Phi+\tfrac{1}{2}\Psi(x),$$

$$\Psi(x) \leq 2\ (N+M_{1,\ 0}+M'\Phi).$$

Se perciò indichiamo con H un numero maggiore di Φ e maggiore anche di $2(N+M_{1,\ 0}+M'\Phi)$, abbiamo, per $0 \leq x < \lambda$,

$$|\ \phi^*(x)\ | < H,$$

e dalla (E) segue, in virtù di (C'_2), che la $\phi^*(x)$ è finita e continua anche per $x=\lambda$. Se ne deduce che, se fosse $\lambda < 1$, la $\phi^*(x)$ esisterebbe, finita e continua, anche al di là di λ, contro la definizione stessa di λ.

8. Ritorniamo al caso di a e b finiti, e supponiamo che valgano la $a < f(o) < b$ e le condizioni (I) e (III) del n°1, e che, invece della (II), sussista la condizione seguente:

(V) Per ogni x di (0, 1), per qualsiasi terna y_1, y_2, y_3, con $0 \leq y_1 \leq y_2 \leq y_3 \leq 1$, e per qualunque funzione $\phi(y)$ definita e continua in (y_1, y_3), ed ivi sempre soddisfacente alla $a \leq \phi(y) \leq b$, sia

$$(C_5) \qquad A\left[x,\ \phi_{y_1}^{y_3}(y)\right] = A\left[x,\ \phi_{y_1}^{y_2}(y)\right] + A\left[x,\ \phi_{y_2}^{y_3}(y)\right],$$

Osserviamo che dalle condizioni (I) e (V) segue che è verificata anche la (II).

Sotte le ipotesi qui poste, ci proponiamo di mostrare che l'esistenza della soluzione dell'equazione (E) può anche ottenersi con un metodo che è la naturale estensione, alla (E), di quello usato da Cauchy per le equazioni differenziali ordinarie.

A questo scopo, dobbiamo premettere un'osservazione.

Consideriamo un punto x di $(0, 1)$, un intervallo (y_1, y_2) pure di $(0, 1)$, ed una funzione $\phi(y)$, data e continua in (y_1, y_2) ed ivi sempre soddisfacente alla limitazione $a \leq \phi(y) \leq b$. Suddiviso l'intervallo (y_1, y_2) in n parti qualsiasi, mediante i punti $y^{(0)}=y_1 < y^{(1)} < y^{(2)} < \dots < y^{(n)}=y_2$, al somma

$$\sum_{r=0}^{n-1} A\left[x, \; \phi_{y^{(r)}}^{y^{(r+1)}} (y^{(r)})\right]$$

tende a

$$A\left[x, \; \phi_{y_1}^{y_2}(y)\right],$$

al tendere di n all'infinito, in modo che tutte le differenze $y^{(r+1)} - y^{(r)}$ tendano a 0. Ed infatti, da (C_5) segue

$$A\left[x, \; \phi_{y_1}^{y_2}(y)\right] = \sum_{r=0}^{n-1} A\left[x, \; \phi_{y^{(r)}}^{y^{(r+1)}} (y)\right].$$

Ma, preso $\epsilon > 0$ ad arbitrio, se tutte le parti $(y^{(r)}, y^{(r+1)})$, in cui si è suddiviso (y_1, y_2), sono tali che in ciascuna di esse l'oscillazione della $\phi(y)$ risulti minore del numero ρ corrispondente ad ϵ secondo la condizione (III), si ha, per la (C_3),

$$\left| A \left[x,\ \phi_{y^{(r)}}^{y^{(r+1)}} (y) \right] - A \left[x,\ \phi_{y^{(r)}}^{y^{(r+1)}} (y^{(r)}) \right] \right|$$

$$< \epsilon\, (y^{(r+1)} - y^{(r)}).$$

È dunque, se tutte le differenze $y^{(r+1)} - y^{(r)}$ sono sufficientemente piccole,

$$\left| A \left[x,\ \phi_{y_1}^{y_2}(y) \right] - \sum_{r=0}^{n-1} A \left[x,\ \phi_{y^{(r)}}^{y^{(r+1)}} (y^{(r)}) \right] \right|$$

$$< \epsilon\, (y_2 - y_1) \leq \epsilon ;$$

e siccome ϵ è arbitrario, la nostra affermazione è provata.

Ciò premesso, dividiamo l'intervallo $(0, 1)$ in n parti uguali, mediante i punti

$$0,\ \frac{1}{n},\ \frac{2}{n},\ \ldots\ldots,\ \frac{n}{n} = 1,$$

e poniamo

$$\psi_n(0) = f(0),$$

$$\psi_n(x) = f(x) + A \left[x,\ \psi_{n\,0}^{\,x}(0) \right], \qquad \text{per} \quad 0 < x \leq \frac{1}{n},$$

$$\psi_n(x) = f(x) + A \left[x,\ \psi_{n\,0}^{\,1:n}(0) \right] + A \left[x,\ \psi_{n\,1:n}^{\,x}\left(\frac{1}{n}\right) \right],$$

$$\text{per} \quad \frac{1}{n} < x \leq \frac{2}{n},$$

$$\psi_n(x) = f(x) + \mathrm{A}\left[x, \psi_{n_0}^{1:n}(0)\right] + \mathrm{A}\left[x, \psi_{n_{1:n}}^{2:n}\left(\frac{1}{n}\right)\right]$$

$$+ \mathrm{A}\left[x, \psi_{n_{2:n}}^{x}\left(\frac{2}{n}\right)\right], \qquad \text{per} \quad \frac{2}{n} < x \leq \frac{3}{n},$$

e così via, sino a definire la $\psi_n(x)$ in tutto l'intervallo $\left(0, \frac{m}{n}\right)$, m essendo il massimo intero tale che, in tutti i punti $0, \frac{1}{n}, \frac{2}{n}, \ldots\ldots, \frac{m-1}{n}$, la $\psi_n(x)$ risulti compresa frr a e b.

Conservando le notazioni del n°2 e con gli stessi ragionamenti ivi fatti, si vede che è $\frac{m}{n} > l$ e che nell'intervallo $(0, l)$ le $\psi_n(x)$ sono tutte comprese fra a e b e tutte ugualmente continue. Esiste quindi almeno una funzione limite continua $\psi_\infty(x)$, per la successione $\psi_1(x), \psi_2(x), \ldots\ldots$, in tutto l'intervallo $(0, l)$.

È facile mostrare che la $\psi_\infty(x)$ è una soluzione dell'equazione (E). Ed invero, indichiamo con $\psi_{n_1}(x), \psi_{n_2}(x), \ldots\ldots, \psi_{n_m}(x), \ldots$ una successione estratta da quella delle $\psi_n(x)$ e convergente uniformemente in $(0, l)$ verso la $\psi_\infty(x)$; poi, preso ad arbitrio un $\epsilon > 0$ consideriamo il numero ρ che ad esso corrisponde seconde la condizione (III) del n°1, e indichiamo con m_1 un intero tale che, per $m > m_1$, sia, in $(0, l)$,

$$|\psi_\infty(x) - \psi_{n_m}(x)| < \rho. \tag{7}$$

Fissato un x di $(0, l)$, possiamo supporre m_1 sufficientemente grande affinchè per ogni $m > m_1$, diviso l'intervallo $(0, 1)$ in n_m parti uguali mediante i punti

$$0, \frac{1}{n_m}, \frac{2}{n_m}, \ldots\ldots, \frac{n_m}{n_m} = 1,$$

sia

$$(8)\qquad \left| A\left[x,\ \psi^{x}_{\infty_0}(y)\right] - \left\{ \sum_{r=0}^{n'-1} A\left[x,\ \phi^{(r+1):n_m}_{\infty_{r:n_m}}\left(\frac{r}{n_m}\right)\right] \right.\right.$$

$$\left.\left. + A\left[x,\ \psi^{x}_{\infty_{n':n_m}}\left(\frac{n'}{n_m}\right)\right]\right\} \right| < \epsilon,$$

dove è

$$\frac{n'}{n_m} \leq x < \frac{n'+1}{n_m}.$$

Ciò è possibile in virtù dell'osservazione fatta in principio di questo n°.

Per ogni $m > m_1$, avremo anche, in forza di (C_3),

$$(9)\qquad \left| \left\{ \sum_{r=0}^{n'-1} A\left[x,\ \psi^{(r+1):n_m}_{\infty_{r:n_m}}\left(\frac{r}{n_m}\right)\right] \right.\right.$$

$$\left. + A\left[x,\ \psi^{x}_{\infty_{n':n_m}}\left(\frac{n'}{n_m}\right)\right]\right\}$$

$$- \left\{ \sum_{r=0}^{n'-1} A\left[x,\ \psi^{(r+1):n_m}_{n_{m_{r:n_m}}}\left(\frac{r}{n_m}\right)\right] \right.$$

$$\left.\left. + A\left[x,\ \psi^{x}_{n_{m_{n':n_m}}}\left(\frac{n'}{n_m}\right)\right]\right\} \right| \leq \epsilon x \leq \epsilon.$$

Risulta, pertanto, per ogni $m > m_1$, tenendo presente la definizione di $\psi_n(x)$,

$$|\ \psi_{\infty}(x) - f(x) - A[x,\ \psi^{x}_{\infty_0}(y)\ | \leq |\ \psi_{\infty}(x) - \psi_{n_m}(x)\ |$$

$$+ \left| \left\{ \sum_{r=0}^{n'-1} A\left[x,\ \psi^{(r+1):n_m}_{n_{m_{r:n_m}}}\left(\frac{r}{n_m}\right)\right] \right.\right.$$

$$+ A\left[x, \psi^{\infty}_{n_{m_{n'}:n_m}}\left(\frac{n'}{n_m}\right)\right]\Big\} - A\left[x, \psi^{*}_{\infty_0}(y)\right] ,$$

e tenendo conto delle (7), (8) e (9),

$$|\ \psi_{\infty}(x) - f(x) - A[x, \psi^{*}_{\infty_0}(y)]\ | < \rho + 2\epsilon < 3\epsilon,$$

perchè possiamo sempre supporre $\rho < \epsilon$ Essendo ϵ arbitrario, ne viene, come appunto volevamo,

$$\psi_{\infty}(x) - f(x) - A[x, \psi^{*}_{\infty_0}(y)] = 0.$$

Ragionando come nel $n^\circ 6$, si vede che, se si suppone verificata anche la condizione (IV) ($n^\circ 4$), e se esiste in tutto un intervallo $(0, \lambda)$ di $(0, 1)$, una soluzione $\phi^*(x)$ dell' equazione (E), sempre soddisfacente alle disuguaglianze $a < \phi^*(x) < b$, la funzione $\psi_n(x)$ definita nel presente n°, e considerata per n sufficientemente grande, converge uniformemente, in tutto $(0, \lambda)$, verso la $\phi^*(x)$.

ÜBER EINEN S. BERNSTEIN-SCHEN SATZ ÜBER DIE DERIVIERTE EINES TRIGONOMETRISCHEN POLYNOMS, UND ÜBER DIE SZEGÖSCHE VERSCHÄRFUNG DESSELBEN.

VON

LEOPOLD FEJÉR IN BUDAPEST.

[*Read July 29, 1928*]

§1. *Die S. Bersteinsche Ungleichung.*

1. Herr **S. Bernstein** hat im Jahre 1912 den folgenden merkwürdigen Satz[1] veröffentlicht.

Ist

$$(1)\quad |\phi(\theta)| = |a_0+a_1\cos\theta+b_1\sin\theta+\ldots+a_n\cos n\theta+b_n\sin n\theta| \leqq 1$$

für jeden reellen Wert von θ, so ist

$$(2)\quad |\phi'(\theta)| \leqq 2n$$

für jeden reellen Wert von θ.

Hier bezeichnen $a_0, a_1, b_1, \ldots a_n, b_n$ reelle Zahlen.

Es ist sogar[2], im wesentlichen wieder nach Herrn **S. Bernstein**,

$$(3)\quad |\phi'(\theta)| \leqq n,$$

aber ich möchte mich in diesen Zeilen nur mit dem Beweise der weniger scharfen Ungleichung (2) beschäftigen, die für gewisse Anwendungen schon

[1] **S. Bernstein 1**, S. 20.

[2] **Fejér 2**, S, 50., Fussnote**); **Fekete 1**. S. 88,89, No. 1.; **M.Riesz, 1**, Fussnote; **2**., S, 356, Fussnote; **F. Riesz 1**; **De la Vallée Poussin 1**. S. 39-42, No. 30; **Pólya-Szegö 1**, Bd. II. S. 287, Aufgabe 82; **S. Bernstein 2**, S. 39,

ausreicht. Namentlich möchte ich hier zeigen, dass die S. Bernstein*sche Ungleichung* (2) *unmittelbar aus dem Satze*[8] *folgt, nach welchem der Wert der arithmetischen Mittel der Partialsummen eines trigonometrischen Polynoms* (*oder einer unendlichen Fourierreihe*) *zwischen der oberen und unteren Grenze der entwickelten Funktion liegt.* Dieser Beweis, und der im § 2, vorkommende, ist, wegen seiner Kürze, vielleicht auch für Vorlesungszwecke geeignet.

2. Es genügt bekanntlich die Ungleichung (2) für ein reines Sinuspolynom, und für die Stelle $\theta=0$ zu beweisen. Dies soll übrigens im No. 3 dargetan werden, zuerst soll aber die Ungleichung (2) für diesen speziellen Fall bewiesen werden.

Es sei

$$\phi(\theta)=b_1 \sin\theta+\ldots+b_{n-1}\sin(n-1)\theta+b_n \sin n\theta. \tag{4}$$

Dann ist

$$2\sin n\theta\phi(\theta)=b_n+b_{n-1}\cos\theta+b_{n-2}\cos 2\theta+\ldots$$
$$+b_1\cos(n-1)\theta-b_1\cos(n+1)\theta-\ldots-b_n\cos 2n\theta. \tag{5}$$

Wird also $|\phi(\theta)|\leqq 1$ für jedes reelle θ vorausgesetzt, so ist

$$|2\sin n\theta\phi(\theta)|\leqq 2,$$

also ist, nach dem zitiertem Satze, das arithmetische Mittel erster Ordnung mit dem Index $n-1$ des Polynoms (5) an der Stelle $\theta=0$ ebenfalls $\leqq 2$ dem absolutem Betrage nach, d.h. es ist

$$\left|\frac{nb_n+(n-1)b_{n-1}+\ldots+1.\,b_1}{n}\right|\leqq 2, \tag{6}$$

d.h.

$$|\phi'(0)|\leqq 2n, \tag{7}$$

womit der in's Auge gefasste Spezialfall der Ungleichung (2) schon bewiesen ist.

[8] Fejér 1, S. 60, Dieser Satz folgt wieder unmittelbar aus der mannigfach elementar beweisbaren Tatsache, dass

$$(n+1)+n\,.\,2\cos\theta+(n-1).\;2\cos 2\theta+\ldots+2\cos n\theta\geqq 0$$

für jeden reellen Wert von θ.

3. Bezeichnet jetzt $\phi(\theta)$ ein beliebiges trigonometrisches Polynom n-ter Ordnung, und θ eine beliebige, aber feste Stelle, so ist

$$\Phi(t)=\frac{\phi(\theta+t)-\phi(\theta-t)}{2} \tag{8}$$

offenbar ein reines Sinuspolynom n-ter Ordnung in t, und es ist $|\Phi(t)| \leqq 1$ für jeden reellen Wert von t, wenn für dieselben Werte $|(t)| \leqq 1$ vorausgesetzt wird. Die Anwendung der schon bewiesenen Ungleichung (7) auf $\Phi(t)$ liefert aber unmittelbar

$$|\Phi'(o)| = |\phi'(\theta)| \leqq 2n,$$

d.h. die zu beweisende Ungleichung.

§2. *Die Verschärfung der S. Bernsteinschen Ungleichung durch Szegö.*

4. Herr Szegö hat in einer demnächst erscheinenden Arbeit der S. Bernsteinschen Ungleichung die folgende überraschende Verschärfung gegeben.

Ist der *reelle Teil* des Polynoms der komplexen Veränderlichen z mit komplexen Koefficienten

$$g(z)=c_0+c_1z+\ldots+c_nz^n \tag{9}$$

für $|z| \leqq 1$ dem absoluten Betrage nach $\leqq 1$, so ist[4]

$$|g'(z)| \leqq n \tag{10}$$

für $|z| \leqq 1$.

Dieser Szegö-sche Satz besagt, dass aus der Voraussetzung nicht nur (3) sondern sogar

$$\sqrt{(\phi'(\theta))^2+(\psi'(\theta))^2} \leqq n \tag{11}$$

folgt, wo $\psi(\theta)$ das zu $\phi(\theta)$ konjugierte trigonometrische Polynom bezeichnet.

[4] Herr **M. Riesz** beweisst $|g'(z)| \leqq n$ unter der engeren Voraussetzung $|g(z)| \leqq 1$ für $|z| \leqq 1$. S. **M. Riesz 2.**, S. 357. Satz I'. Einen schönen Beweis dieses **M. Riesz**schen Satzes verdankt man Herrn **O. Sza'sz**, der sich ausschliesslich auf die auch in dieser Arbeit benützen Fundamentalungleichung der arithmetischen Mittel stützt, S. **Sza'sz 2**,. S. 516., 517. § 5.

Statt (10) betrachte ich wieder nur die viel leichter beweisbare Ungleichung

$$(11) \qquad |g'(z)| \leqq 2n,$$

und beweise sie in folgendes Weise.

Dafür $z=e^{i\theta}$

$$(13) \quad 2\times \text{reeller Teil von } g(z)=c_0+c_1z+\ldots+c_nz^n+$$

$$+\bar{c}_0+\frac{\bar{c}_1}{z}+\ldots+\frac{\bar{c}_n}{z^n}=$$

$$=\frac{1}{z^n}\left(\bar{c}_n+\bar{c}_{n-1}z+\ldots+\bar{c}_1z^{n-1}+(\bar{c}_0+c_0)z^n+\cdots+c_nz^{2n}\right),$$

so ist, mit Rücksicht auf die Voraussetzung, das Polynom $2n$-ten Grades

$$(14) \qquad \bar{c}_n+\bar{c}_{n-1}z+\ldots+\bar{c}_1z^{n-1}+\ldots+c_nz^{2n}$$

für $|z|=1$ dem absoluten Betrage nach $\leqq 2$. Also gilt auch für das arithmetische Mittel erster Ordnung mit dem Index $n-1$ des Polynoms (14), an der Stelle $z=1$,

$$(15) \qquad \left|\frac{n\bar{c}_n+(n-1)\bar{c}_{n-1}+\ldots+\bar{c}_1}{n}\right|\leqq 2,$$

d.h.

$$(16) \qquad |nc_n+(n-1)c_{n-1}+\ldots+c_1|=|g'(1)|\leqq 2,$$

womit die Ungleichung (12) für $z=1$ bewiesen ist.

Um sie für eine beliebige andere Stelle ζ der Einheitskreislinie zu beweisen, wollen wir das Polynom $g(\zeta z)$ betrachten. Der reelle Teil dieses Polynoms in z ist für $|z|\leqq 1$ dem absoluten Betrage nach $\leqq 1$. Also ist auf Grund von (16), tatsächlich

$$(17) \qquad \left|\frac{d}{dz}g(\zeta z)\right|_{z=1}=\left|\zeta g'(\zeta z)\right|_{z=1}=\left|\zeta g'(\zeta)\right|=\left|g'(\zeta)\right|\leqq 2n,$$

woraus dann $|g'(\zeta)|\leqq 2n$ auch für $|\zeta|<1$ folgt.

Budapest, den 20-ten Juni 1928.

LEOPOLD FEJÉR

LITTERATURVERZEICHNIS.

Serge Bernstein.

1. "Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné," (mémoire couronné par la Classe des sciences), *Mémoires publiés par la Classe des sciences de l'Académie royale de Belgique* deuxième série, tome IV. 1912.

2. *Lecons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle.* Paris, 1926.

L. Fejér.

1. "Untersuchungen über Fouriersche Reihen," *Math. Annalen* Bd. 58 (1904), S. 51-69.

2. "Über conjugierte trigonometrische Reihen," *Journal für die reine und angewandte Mathematik,* Bd. 144, (1914), S. 48-56.

M. Fekete.

1. "Über einen Satz des Herrn Serge Bernstein," *Journal für die reine und angewandte Mathematik,* Bd. 146. S. 88-94.

G. Pólya und **G. Szegö.**

1. "Aufgaben und Lehrsätze aus der Analysis," Bd. II. Berlin, 1925.

Marcel Riesz.

1. "Formule d'interpolation pour la dérivée d'un polynome trigonométrique, *Comptes-Rendus,* t. 158. (1914).

2. "Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome."

Friedrich Riesz.

1. "Sur les polynomes trigonométriques," *Comptes-Rendus,* t. 158. (1914).

O. Sza'sz.

1. "Ungleichungen für die Koeffizienten einer Potenzreihe," *Math. Zeitschrift.*, Bd. 1. (1918) S. 163-183.

2. "Über beschränkte Potenzreihen" (ungarisch), *Mathematischer und naturwissenschaftlicher Anzeiger der Ungarischen Akademie der Wissenschaften*, Bd. XLIII, (1926) S. 504-520.

De la Valle'e Poussin.

1. *Leçons sur l'approximation des fonctions d'une variable réelle.* Paris, 1919.

SUR L'APPROXIMATION DES FONCTIONS CONTINUES ET DES FONCTIONS SOMMABLES

Par

Frédéric Riesz à Szeged (*Hongrie*).

[*Communicated by Professor Ganesh Prasad, July 29, 1928.*]

1. Soient E un ensemble mesurable, de mesure positive p un nombre > 1 et désignons par L^p la classe des fonctions sommables sur cet ensemble ainsi que la $p^{\text{ième}}$ puissance de leur module. Soit L^q la classe analogue correspondant à l'exposant q tel que $1/p + 1/q = 1$ c'est-à-dire à $q = p/p-1$.

J'ai établi, il y a presque vingt ans, le théorème suivant:

Théorème *A. Pour que le système d'un nombre fini ou d'une infinité dénombrable d'équations intégrales*

$$(1) \qquad \int_E f_k(x)\phi(x)dx = c_k \qquad (k=0, 1, 2, \ldots),$$

où les fonctions données $f_k(x)$ appartiennent à la classe L^p, admette une solution $\phi(x)$ appartenant à la classe L^q et telle que

$$(2) \qquad \int_E |\phi(x)|^q dx \leq M^q,$$

il faut et il suffit que l'inégalité

$$(3) \qquad \left| \sum_{k=0}^{n} \mu_k c_k \right| \leq M \left[\int_E \left| \sum_{k=0}^{n} \mu_k f_k(x) \right|^p dx \right]^{1/p}$$

ait lieu pour toute valeur de n et des μ_k.[1]

[1] *Untersuchungen über Systeme integrierbarer Funktionen*, Math. Annalen 69-1910, pp. 449-497.

Peu de temps après, j'en ai déduit un second théorème, correspondant au cas limite $p \longrightarrow \infty$, $q \longrightarrow 1$, mais que l'on peut aussi établir par une voie directe. Le voilà :

Théorème B. *Pour que le système d'un nombre fini ou d'une infinité dénombrable d' équations intégrales*

$$\int_a^b f_k(x)d\alpha(x)=c_k \qquad (k=0,1,2,\ldots)$$

où les fonctions données $f_k(x)$ sont supposées d'être continues, la fonction inconnue $\alpha(x)$ est soumise à la condition d'être à variation bornée et les intégrales sont prises au sens de Stieltjes, *admette une solution $\alpha(x)$ dont la variation totale, sur l'intervalle ab, ne surpasse pas une certaine borne M, il faut et il suffit que l'inégalité*

$$\left| \sum_{k=0}^{n} \mu_k c_k \right| \leq \mathrm{M} \times \max_{a \leq x \leq b} \left| \sum_{k=0}^{n} \mu_k f_k(x) \right|$$

ait lieu pour toute valeur de n et des μ_k.[1]

La Note présente a pour but d'attirer l'attention à quelques conséquences presque immédiates de ces théorèmes ; malgré leur évidence, ces conséquences paraissent être restées inaperçues pendant tout le temps.

2. Soit $f_1(x)$, $f_2(x)$,... une suite d'éléments de la classe L^p et supposons que cette suite converge *faiblement*, d'ordre p, vers la fonction $f(x)$, appartenant elle-même à la classe L^p, Cela veut dire que l'on a, pour toute fonction $\phi(x)$ appartenant à la classe L^q,

$$\int_E f_n(x)\phi(x)dx \longrightarrow \int_E f(x)\phi(x)dx. \tag{4}$$

[1] *Sur certaines systèmes d'équations fonctionnelles et l'approximation des fonctions continues*, Comptes rendus, Paris, 14 mars 1910. *Cf.* encore **E. Helly**, *Über lineare Funktionaloperationen*, Sitzungsberichte d. k. Akademie d. Wiss., Wien, *121*-II *a*-1912, pp. 265-297.

On sait bien que, dans l'étude des classes L^p, on se sert, outre l'idée de la convergence faible, de celle de la convergence *forte* ou convergence *en moyenne* d'ordre p, caractérisée par la relation

$$\int_E |f(x)-f_n(x)|^p dx \longrightarrow 0.$$

On sait aussi que la convergence forte implique la convergence faible tandis que la réciproque n'est pas vraie.[1] Cependant, nous allons montrer qu'il est toujours possible de passer de la convergence faible à la convergence forte et cela en remplaçant les éléments $f_n(x)$ de la suite par des combinaisons linéaires convenablement choisies. D'une façon précise, nous allons établir le théorème suivant :

Théorème *C. Lorsque la suite* $\{f_n(x)\}$, *formée d'éléments de la classe* L^p, *converge faiblement, d'ordre p, vers la fonction* $f(x)$, *on peut déterminer l'entier n et les coefficients* $\mu_1, \mu_2, \ldots, \mu_n$ *de sorte que la valeur de l'intégrale*

$$(5) \qquad \int_E \left| f(x) - \sum_{k=1}^{n} \mu_k f_k(x) \right|^p dx$$

devienne arbitrairement petite.

En effet, supposons, par impossible, que la valeur de l'intégrale (5) reste supérieure, pour tout choix de n et des μ_k, à la $p^{\text{ième}}$ puissance d'un nombre $\delta > 0$. Cela étant, envisageons le système d'équations (1) avec

$$f_0(x)=f(x),\ c_0=1,\ c_k=0\ (k>0).$$

Alors l'hypothèse concernant l'intégrale (4) que nous venons de faire est équivalente, par raison d'homogénéité, à l'hypothèse (3), avec $M=1/\delta$. Or, le théorème *A* nous assure que, sous cette hypothèse, le système (1) admet une solution telle que (2). C'est-à-dire qu'il existe une fonction $\phi(x)$ appartenant à la classe L^q et telle que

$$\int_E f(x)\phi(x)dx=1, \quad \int_E f_k(x)\phi(x)dx=0 \quad (k=1, 2, \ldots),$$

ce qui est en contradiction avec la relation (4).

Le théorème *C* est donc démontré.

3. En partant du théorème B, un raisonnement analogue à celui qui précède nous conduira à un théorème concernant les suites de fonctions continues.

Théorème D. *Lorsque la suite* $\{f_n(x)\}$, *formée de fonctions continues pour* $a \leq x \leq b$, *converge partout vers la fonction continue* $f(x)$ *et que, de plus, les fonctions* $f_n(x)$ *restent bornées dans leur ensemble, alors il existe une suite de combinaisons linéaires des fonctions* $f_n(x)$ *qui tend uniformément vers la fonction* $f(x)$.

En effet, dans l'hypothèse contraire, on pourrait affirmer, d'après le théorème B et comme je l'ai déjà observé dans ma Note citée,[2] l'existence d'une fonction à variation bornée $\alpha(x)$ de sorte que

$$\int_a^b f(x)d\alpha(x)=1\,; \quad \int_a^b f_k(x)d\alpha(x)=0 \quad (k=1,2,\ldots).$$

D'autre part, d'après le théorème fondamental de M. Lebesgue sur l'intégration terme à terme des suites bornées et qui s'applique aussi au cas des intégrales de Stieltjes,[3] on devrait avoir

$$0=\int_a^b f_k(x)d\alpha(x) \longrightarrow \int_a^b f(x)d\alpha(x)=1,$$

ce qui implique contradiction.

4. Nos résultats s'étendent sans peine aux fonctions de plusieurs variables et, quant aux théorèmes B et D, au cas d'un ensemble fermé quelconque.

Szeged, le 6 mai 1928.

[3] *Cf.* par ex. ma Note : *Sur le théorème de M. Egoroff et sur les opérations fonctionnelles linéaires*, Acta Universitatis Franc.—Jos., Szeged, *1*-1922. pp. 18-26.

ON THE THEORY OF INDETERMINATE EQUATIONS OF THE SECOND DEGREE IN TWO VARIABLES

(An ideal-theoretic exposition)

BY

Teiji Takagi (Tokyo)

(*Read August 19, 1928*)

[Communicated by Professor Ganesh Prasad]

The indeterminate equation of the second degree with two unknowns, which has been studied with remarkable success by the mathematicians of ancient India and which still occupies a prominent place in the elementary theory of numbers, has nevertheless received little attention from the writers of the modern treatises of the subject, most of them satisfying themselves with a very summary reference of the problem to the classical method of Gauss. In the following lines we give a synoptic treatment of the problem from the ideal-theoretic point of view.

We denote by *italics* rational integers, by Greek letters numbers, mostly integers, belonging to a quadratic corpus $K=K(\sqrt{d})$, where d is the discriminant of the corpus K, so that $d\equiv 0$ or 1 (mod. 4), being at the same time free from redundant square factors. In the equation

$$ax^2+bx+c=k \qquad \dots \quad (1)$$

* For the elements of the theory of the quadratic corpus, as are here referred to, the following works may be consulted :

Reid, *The Elements of the Theory of Algebraic Numbers.*

Sommer, *Einführung in die Theorie der algebraischen Zahlen.*

Bachmann, *Grundlehren der neueren Zahlentheorie.*

Hecke, *Theorie der algebraischen Zahlen.*

Landau, *Vorlesungen über Zahlentheorie*, 3.

we suppose a, b, c freed from common divisors, $a>0$, and the discriminant

$$D=b^2-4ac=Q^2d$$

not a perfect square. Then if we put

$$\theta=\frac{b+\sqrt{d}}{2},$$

(1) is equivalent to

$$\mathrm{N}(ax+\theta y)=ak, \qquad \dots \quad (1^*)$$

N denoting the norm taken in K, *i.e.*, $\mathrm{N}\xi=\xi\xi'$, where ξ,ξ' are the conjugates in K.

Denote by $\mathfrak{a}$ the ideal

$$\mathfrak{a}=(a,\theta),$$

then

$$\mathrm{N}\,\mathfrak{a}=(a,\theta)(a,\theta')=(a^2,\,a\theta,\,a\theta',\,\theta\theta')=a(a,\,b,\,c)=a.$$

If (1^*) admits of a solution

$$\alpha=ax+\theta y,$$

then, α being divisible by $\mathfrak{a}$, we may put

$$\alpha=\mathfrak{a}\mathfrak{k}.$$

Then

$$\mathrm{N}\,\mathfrak{a}\mathfrak{k}=|\,\mathrm{N}\alpha\,|=a\,|\,k\,|,$$

so that

$$\mathrm{N}\,\mathfrak{k}=|\,k\,|.$$

Conversely, if $|\,k\,|$ is the norm of an ideal $\mathfrak{k}$ of K, such that

$$\mathfrak{a}\mathfrak{k}=(\alpha)$$

is a principal ideal generated by a number α, whose norm has the same sign as k, then if

(Case I) $\qquad D=d,$

all the solutions of (1^*) will be given by

$$ax+\theta y=\alpha\epsilon, \qquad \mathrm{N}\epsilon=1,$$

where ϵ is a unity of K, since in this case $[a,\theta]$ being a canonical basis of $\mathfrak{a}$, any number $\alpha\epsilon$, which is divisible by $\mathfrak{a}$ is necessarily of the form $ax+\theta y$.

Thus the solution of (1*) is reduced to the following problems:

(I) Given a positive rational integer k, to find all the ideals $\mathfrak{k}$ of K, such that

$$N\mathfrak{k}=k.$$

(II) Given an ideal $\mathfrak{j}(=\mathfrak{a}\mathfrak{k})$ of K, to determine if $\mathfrak{j}$ is a principal ideal (α) and to find a value of α.

(III) To find the fundamental unity of the real quadratic corpus $K(\sqrt{d})$.

Again if

(Case II) $\qquad D\neq d$, *i.e.*, $\quad Q>1$,

then (a, θ) is not a basis of $\mathfrak{a}$, consequently the numbers $\alpha\epsilon$ are not necessarily of the required form $ax+\theta y$. But if $D<0$, then, since to each $\mathfrak{k}$ can correspond only a pair $\pm\alpha$, we have no difficulty in seeing if they are or not of the form $ax+\theta y$.

If, however, $D>0$, then with α correspond the whole system of the associates $\alpha\epsilon$ to an ideal $\mathfrak{k}$, so that we are confronted with the problem:

(IV) Given a number α, such that $N\alpha=ak$, to pick out from the system of the associates $\alpha\epsilon$ those of the form $ax+\theta y$.

Problem I. If, in canonical representation

$$\mathfrak{k}=s\left[k_0, \frac{r+\sqrt{d}}{2}\right],$$

then

$$N\left(\frac{r+\sqrt{d}}{2}\right)\equiv 0 \pmod{k_0}$$

and

$$N\mathfrak{k}=s^2k_0.$$

Hence the problem will be completely solved, if we find all the solutions of the congruence

$$r^2\equiv d \pmod{4k_0}$$

which are incongruent mod. $2k_0$.

If we distinguish between the three kinds of rational primes with respect to $K(\sqrt{d})$:

$$p=\boldsymbol{p}\boldsymbol{p}',\ (\boldsymbol{p}\neq\boldsymbol{p}'),\ q=\boldsymbol{q},\ l=\boldsymbol{l}^2$$

and put

$$k=p^a\dots q^b\dots l^c\dots$$

$$\boldsymbol{k}=\boldsymbol{p}^{\alpha}\boldsymbol{p}'^{\alpha'}\dots\boldsymbol{q}^{\beta}\dots\boldsymbol{l}^{\gamma}\dots$$

then

$$N\boldsymbol{k}=k,$$

when

$$\alpha+\alpha'=a,\ 2\beta=b,\ \gamma=c.$$

In order that the problem may admit of a solution, it is therefore necessary that k contains primes of the second kind in even powers ; and if this condition is satisfied, all the solutions are given by $\boldsymbol{k}=s\,\boldsymbol{k}_0$, where

$$s=p^u\dots q^{\frac{b}{2}}\dots l^{c_0}\quad\dots\quad \boldsymbol{k}_0=\boldsymbol{p}^{a_0}\dots\boldsymbol{l}^{\gamma_0}.$$

$$(u=0,\ 1,\dots,\ [\tfrac{a}{2}]\ ;\ a_0=a-2u\ ;\ c=2c_0+\gamma_0,\ \gamma_0=0,\ 1.)$$

each $\boldsymbol{p}$ being also replaced by $\boldsymbol{p}'$. The total number of the solutions amounts to $\prod 2(1+[\frac{a}{2}])$.

Problem II. We may assume $\boldsymbol{j}$ freed from rational divisors, and given in canonical form :

$$\boldsymbol{j}=[a,\ \frac{b+\sqrt{d}}{2}].$$

Let $[1,\ \omega]$ be a basis of the integers of the corpus $\mathbf{K}$.

Also put

$$\Theta=\frac{b+\sqrt{d}}{2},\ \omega=\frac{r+\sqrt{d}}{2}. \qquad \dots\ (2)$$

Then if

$$\mathfrak{j}=(\alpha), \qquad \dots \quad (3)$$

or

$$[a, a\Theta]=[\alpha, \alpha\omega],$$

the two systems of bases are connected by modular substitutions :

$$a\Theta=\alpha(p\omega+p'),$$

$$a=\alpha(q\omega+q'), \qquad \dots \quad (4)$$

$$pq'-p'q=e=\pm 1.$$

Hence

$$\Theta=\frac{p\omega+p'}{q\omega+q'}. \qquad \dots \quad (5)$$

Conversely, if (5) subsist, we may set (4) and get back to (3).

Now from (4) we get

$$\begin{vmatrix} a\Theta & a\Theta' \\ a & a \end{vmatrix} = \begin{vmatrix} \alpha\omega & \alpha \\ \alpha'\omega' & \alpha' \end{vmatrix} \begin{vmatrix} p & p' \\ q & q' \end{vmatrix}$$

or

$$\mathrm{N}\alpha=ea=\pm a.$$

so that by (4)

$$\alpha=\pm(q\omega'+q'). \qquad \dots \quad (6)$$

When $\mathbf{K}$ is real, it is well known, how to find (5) by means of continued fractions. In this case it is most convenient to take r in (2) so that ω becomes the so-called reduced irrational $(\omega>1, 0>\omega'>-1)$, r is then the greatest integer not exceeding $\sqrt{d}$ and $\equiv d$ (mod. 2). Should (5) subsist, the expansion of Θ into a continued fraction must yield ω as a complete quotient, and when this continued fraction is reduced to the form (5), we get α from the denominator by (6) *The numerator need not be calculated.*

When K is imaginary, it is advisable to take r in (2) so that ω may fall in the usual discontinuity-domain of modular substitutions : $|\omega|\geq 1$, $\frac{1}{2}> R(\omega)\geq -\frac{1}{2}$, then (5) can be easily determined by the classical method.

Problem III is very well-known. We beg a little patience of the reader to give a simplified demonstration. Let Θ be a reduced irrational, *e.g.*, $\Theta=\frac{r+\sqrt{d}}{2}$ as above. The continued fraction for Θ is then purely periodic. Taking any number of periods we reduce it to the form

$$\Theta=\frac{p\Theta+p'}{q\Theta+q'} \qquad \dots \quad (7)$$

where

$$pq'-p'q=\pm 1.$$

We can then put

$$\Theta\epsilon=p\Theta+p',$$

$$\epsilon=q\Theta+q',$$

whence

$$\begin{vmatrix} p-\epsilon & p' \\ q & q'-\epsilon \end{vmatrix}=0,$$

or

$$\epsilon^2-(p+q')\epsilon\pm 1=0.$$

Thus from the denominator of (7) we get a unity

$$\epsilon=q\omega+q'>1,\ N\epsilon=\pm 1,$$

the numerator not needed.

Conversely, let ϵ be any unity >1. Evidently ϵ is of the form $x\Theta+y$ with rational integral x, y and consequently $\epsilon\Theta$ also. We can therefore put

$$\epsilon\Theta=p\Theta+p',\quad \epsilon=q\Theta+q',\quad \Theta=\frac{p\Theta+p'}{q\Theta+q'} \qquad \dots \quad (8)$$

and hence

$$\begin{vmatrix} p-\epsilon & p' \\ q & q'-\epsilon \end{vmatrix}=0,$$

or

$$\epsilon^2-(p+q')\epsilon+(pq'-p'q)=0,$$

showing that

$$pq'-p'q=\mathrm{N}\epsilon=\pm 1.$$

Now since by supposition $\epsilon>1>\epsilon'$, we have $q\Theta+q'>q\Theta'+q'$ or $q>0$. Hence from $0>\Theta'>-1$,

$$q'>\epsilon'>-q+q'.$$

Hence, if $\mathrm{N}\epsilon=1$, $1>\epsilon'>0$, we have $q'>0$, $1>-q+q'$ or $q\geq q'$, that is, $q\geq q'>0$. Again if $\mathrm{N}\epsilon=-1$, $0>\epsilon'>-1$, we have $q'\geq 0$, $0>-q+q'$, or $q>q'$, that is, $q>q'\geq 0$. These relations for q, q' imply that (8) surely arises from the continued fraction for Θ. If especially a single period of the expansion be taken, we get the fundamental unity $E=q\Theta+q'=\frac{T+U\sqrt{d}}{2}$ corresponding to the smallest positive solution of " Bhâskara's equation " $T^2-dU=\pm 4$.

The unities of the form $\frac{t+u\sqrt{D}}{2}$, $D=Q^2d$, can be found in a similar way. Generally speaking, however, it is more convenient for practice to calculate the successive powers of the fundamental unity E found above until we get for the first time a unity $E^h=E(Q)=\frac{t+u\sqrt{d}}{2}$, where u is divisible by Q. This takes place for a value of h, which divides $\phi(Q)$, ϕ denoting Euler's function in K. Then $E(Q)=\frac{T+U\sqrt{D}}{2}$ is the unity corresponding to the smallest positive solution of $T^2-DU^2=\pm 4$

Problem IV. We observe first that a number of the form $ax+\theta y$, when multiplied by a unity

$$\eta=\frac{t+u\sqrt{D}}{2}$$

always gives a number of the same form. In fact, since $\theta = \frac{b+\sqrt{D}}{2}$, $\eta - u\theta$ is rational and integral, say v, that is

$$\eta = v + u\theta.$$

Hence

$$(ax+\theta y)\eta = ax\eta + \theta^2 yu.$$

The first term on the right is evidently of the required form and the second $\theta^2 yu$ also, since $\theta^2 = -b\theta - ac$.

Thus if $E(Q) = E^h$ as above, we have, to solve the problem completely, only to examine the h numbers

$$a,\ aE\ aE^2, \ldots aE^{h-1}$$

and pick out those of the required form $ax+\theta y$. If there are $w (w \geq 0)$ of them then these multiplied by $\pm E(Q)^n$, $n=0$, ± 1, $\pm 2, \ldots$ will answer the problem.

Without going into details, let it be noticed that if ak is relatively prime to Q, $w=0$ or 1, while in the contrary case it may happen that $w>1$.

A PROBLEM ON PROBABILITY

BY

TSURUICHI HAYASHI (*Sendai, Japan*)

(*Read August 19, 1928*)

[Communicated by Professor Ganesh Prasad]

In the paper bearing the title " Sur quelques points du calcul des probabilités," * Harald Cramér treated the problem :—*Faisons n tirages successives d'une urne renfermant des boules blanches et noires, la probabilité d'amener une boule blanche étant toujours égale à p. Quelle est la probabilité $a_{n,\mu}$ d'avoir dans le cours de ces n tirages une suite de μ tirages ne donnant que des boules blanches ?* He added that the problem was treated previously by De Moivre, Condorcet and Laplace, that the recurrence-formula for it is

$$a_{n,\mu} = a_{n-1,\mu} + p^{\mu} q(1 - a_{n-\mu-1,\mu}), \quad (q=1-p),$$

and that the *fonction génératrice* for it is

$$F_{\mu}(x) = \sum_{n=0}^{\infty} \beta_{n,\mu} x^n = \frac{1-p^{\mu} x^{\mu}}{1-x+p^{\mu} q x^{\mu+1}}, (\beta_{n,\mu} = 1 - a_{n,\mu}).$$

But he did not show any explicit expression of $a_{n,\mu}$.

In the paper bearing the title " Certain mathematical questions suggested by the true-false test," † H. M. Walker treated the problem, essentially

* Proceedings of the London Mathematical Society, second series, volume 23, 1925, p lxiii.

† The American Mathematical Monthly, volume 34, 1927, p. 508.

similar to the above :—*In a true-false test of n statements, if the order is determined wholly by chance, what is the probability that there will be at least one run of k or more consecutive statements all of which are "true"?* The problem has been enunciated in another way: *An unbiased coin is spun on its edge n times, a plus sign is recorded for each showing of heads and a minus sign is recorded for each showing of tails. What is the probability that there will be at least one run of k or more consecutive plus signs in the record ?* The recurrence-formula

$$p_{n,k}=a^{k+1}+p_{n-1,k}-a^{k+1}\,p_{n-k-1,k}, \quad (a=\tfrac{1}{2})$$

has been given, and the explicit expression of $p_{n,k}$ has been gotten by applying the Calculus of Finite Differences.

If we let $p=\frac{1}{2}$ in the recurrence-formula of Cramér, we get that of Walker, who has stated in his paper that M. H. Stone has deduced the expression of Walker by using the function génératrice shown by Cramér in that particular case.

In this paper I will show an elementary method of getting the explicit expression in the general case of Cramér, without using the idea of the fonction génératrice and without aid of the Calculus of Finite Differences.

The quantity $a_{n,\mu}$ in the case of Cramér and the quantity $p_{n,k}$ in the case of Walker are both some functions of two variables. But either μ or k does not vary in the respective recurrence formula, so that it may be considered as a constant. Moreover the quantity $p^{\mu}q$ in the case of Cramér and the quantity a^{k+1} in the case of Walker may be considered as constants, so that I will replace them by C during the calculation, and restore them by the respective values after the calculation. Then our problem is reduced to :—*To solve the functional equation*

$$f(x)=C+f(x-1)-Cf(x-k-1) \qquad \dots \quad (1)$$

where c and k are some constants, k being a positive integer.

From the nature of the problem it is evident that $f(x)$ is equal to zero when x is less than the positive integer k. So the value of x in equation (1) cannot be less than k. If $x<2k+1$, *i.e.*, $x-k-1<k$, then $f(x-k-1)=0$. Hence while x increases taking positive integral values from k up to $2k+1$, the third term on the right-hand side of equation (1) does not occur.

Thus the equation becomes

$$f(x)=C+f(x-1).$$

Moreover if $x=k$, then $f(x-1)=0$ and obviously $f(k)=p^k$.

Therefore we get the following series successively

$$f(k)=p^k,$$

$$f(k+1)=p^k+C,$$

$$f(k+2)=p^k+2C,$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$f(2k)=p^k+kC.$$

These are the initial conditions which must be satisfied by the required function $f(x)$.

By adding together equation (1) and those equations gotten from (1) by replacing x by $x-1$, $x-2, \ldots\ldots, 2k+1$ successively, we get

$$f(x)=p^k+(x-k)\,C-C\{f(x-k-1)+f(x-k-2)+\ldots\ldots+f(k+1)+f(k)\}.$$

From the last term on the right-hand side of this equation, we see that $f(x)$ is determined if we know the series of the values of the function, going back from that of the argument diminished by $k+1$, *i.e.*, the value of $f(x-k-1)$ and ending at the first value $f(k)$. Therefore we arrange the values of the function in the following groups :—

$$\begin{array}{lllll}
f(k), & f(k+1). & \ldots & \ldots, & f(2k)\,; \\
f(2k+1), & f(2k+2), & \ldots & \ldots, & f(3k+1)\,; \\
f(3k+2), & f(3k+3), & \ldots & \ldots, & f(4k+2) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
f(sk+s-1), & f(sk+s), & \ldots & \ldots, & f(\overline{s+1}\,k+s-1) \\
\ldots & \ldots & & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

each group on one row consisting of $k+1$ members. The tth member on the sth row, *i.e.* $f(sk+s-1+t-1)$, is expressible by adding all the members on the first, second,......$(s-2)$th rows, and the members from the first to the tth on the $(s-1)$th row.

The first term of the required expression of $f(sk+s-1+t-1)$ is obviously

$$p^k+(x-k)C,$$

i.e. $$p^k+\{(s-1)(k+1)+t-1\}C,$$

i.e. $$[p^k+\binom{\sigma}{1}C]_{\sigma}=(s-1)(k+1)+t-1.$$

Hence its second term is

$$-C\sum_{\sigma=0}^{\sigma=(s-2)(k+1)+t-1}[p^k+\binom{\sigma}{1}C],$$

understanding that $\binom{0}{1}=0$. By the well-known theorem in the Combinatory Analysis,

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots\ldots+\binom{n}{r}=\binom{n+1}{r+1}.$$

Hence the second term becomes

$$-C\left[\binom{\sigma}{1}p^k+\binom{\sigma}{2}C\right]_{\sigma=(s-2)(k+1)+t.}$$

Hence the third term is

$$+C^2\sum_{\sigma=1}^{\sigma=(s-3)(k+1)+t}\left[\binom{\sigma}{1}p^k+\binom{\sigma}{2}C\right],$$

understanding that $\binom{1}{2}=0$. Transforming this expression again by applying the same theorem in the Combinatory Analysis, we get

$$+C^2\left[\binom{\sigma}{2}p^k+\binom{\sigma}{3}C\right]_{\sigma=(s-3)(k+1)+t+1.}$$

Similarly we get the fourth term in the form

$$-C^{3}\left[\binom{\sigma}{3}p^{k}+\binom{\sigma}{4}C\right]_{\sigma=(s-4)(k+1)+t+2.}$$

And so forth. Therefore

$$f(sk+s-1+t-1)$$

$$=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}C^{\nu-1}\left[\binom{\sigma}{\nu-1}p^{k}+\binom{\sigma}{\nu}C\right]_{\sigma=(s-\nu)(k+1)+t+\nu-2,}$$

where $\binom{\sigma}{\lambda}=0$ when $\sigma<\lambda$, and $1\leqq t\leqq k+1$.

If we take the notation of Cramér,

$$k=\mu,\quad n=sk+s-1+t-1,\quad 1\quad t\leqq k+1.$$

Hence

$$a_{n,\mu}=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}p^{\mu(\nu-1)}q^{\nu-1}\left[\binom{\sigma}{\nu-1}p^{\mu}\right.$$

$$\left.+\binom{\sigma}{\nu}p^{\mu}q\right]_{\sigma=n-\mu\nu,}$$

or

$$a_{n,\mu}=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}p^{\mu\nu}q^{\nu-1}\left[\binom{\sigma+1}{\nu}-\binom{\sigma}{\nu}p\right]_{\sigma=n-\mu\nu,}$$

where s is determined by the values of n and k satisfying the two relations

$$n=s(\mu+1)+\zeta,\qquad -1\leqq\zeta\leqq\mu-1.$$

If we take the notation of Walker,

$$\mu=k,\quad p=\tfrac{1}{2},\quad q=1-\tfrac{1}{2}=\tfrac{1}{2},\quad p^{\mu}=2p^{\mu}q,$$

$$p^{\mu}q=a^{\mu+1},\quad a=\tfrac{1}{2}.$$

Hence

$$p_{n,k}=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}a^{(k+1)\nu}\left[2\binom{\sigma}{\nu-1}+\binom{\sigma}{\nu}\right]_{\sigma=n-k\nu}$$

$$=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}a^{(k+1)\nu}\left[2\,{}_{\sigma-\nu+2}\mathrm{H}_{\nu-1}+{}_{\sigma-\nu+1}\mathrm{H}_{\nu}\right]_{\sigma=n-k\nu}$$

where ${}_{\sigma}\mathrm{H}_{\nu}$ is the homogeneous product and equal to

$$\frac{\sigma(\sigma+1)(\sigma+2)\ldots\ldots(\sigma+\nu-1)}{\nu!}.$$

Hence by the formula

$${}_{\sigma-1}\mathrm{H}_{\nu}+{}_{\sigma}\mathrm{H}_{\nu-1}={}_{\sigma}\mathrm{H}_{\nu},$$

$$p_{n,k}=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}a^{(k+1)\nu}\left[{}_{\sigma-\nu+2}\mathrm{H}_{\nu-1}+{}_{\sigma-\nu+2}\mathrm{H}_{\nu}\right]_{\sigma=n-k\nu}$$

$$=\sum_{\nu=1}^{\nu=r}(-1)^{\nu-1}a^{(k+1)\nu}$$

$$\times\left[\frac{s(s+1)(s+2)\ldots\ldots(s+\nu-2)(s+2\nu-1)}{\nu!}\right]_{s=n-\nu k-\nu+2}$$

where r is determined by the values of n and k satisfying the two relations

$$n=r(k+1)+\zeta,\quad -1\leqq\zeta\leqq k-1.$$

I have shown that

$$a_{n,\mu}=\sum_{\nu=1}^{\nu=s}\ (-1)^{\nu-1}\ p^{\mu(\nu-1)}\,q^{\nu-1}\left[\binom{\sigma}{\nu-1}p^{\mu}+\binom{\sigma}{\nu}p^{\mu}\,q\right]_{\sigma=n-\mu\nu}$$

in the notation of Cramér If we have this general explicit expression of $a_{n,\ \mu}$, we can easily prove the inequality shown by Cramér which is satisfied by $a_{n,\ \mu}$ for a large value of n. For,

$$a_{n,\ \mu}=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}\ p^{\mu(\nu-1)}\,q^{\nu-1}\left[\ {}_{\zeta+2}\mathrm{H}_{\nu-1}\ p^{\mu}\ +{}_{\zeta+1}\ \mathrm{H}_{\nu}\ p^{\mu}\ q\right]$$

$$=\sum_{\nu=1}^{\nu=s}\ (-1)^{\nu-1}\ \frac{p^{\mu\nu}\,q^{\nu}}{\nu\,!}\ \left\{\frac{\nu}{q}+(\zeta+1)\ \right\}(\zeta+2)(\xi+3)\ldots(\zeta+\nu)$$

$$=\sum_{\nu=1}^{\nu=s}(-1)^{\nu-1}\ \frac{(np^{\mu}\ q)^{\nu}}{\nu\ !}\ .\ \frac{\nu+q(\zeta+1)}{nq}.\ \frac{\zeta+2}{n}.\frac{\zeta+3}{n}\ldots\ldots\frac{\zeta+\nu}{n}.$$

But even for the greatest possible value of ν, we have

$$\zeta+\nu<n\,;$$

and when μ is fixed that is to say, when ζ lies between -1 and $\mu-1$, we have

$$\frac{\nu+q(\zeta+1)}{nq}<\frac{\nu+(\zeta+1)}{nq}\leq\frac{s+(\zeta+1)}{nq}=\frac{\frac{n-\zeta}{\mu+1}+(\zeta+1)}{nq}$$

$$<\frac{1}{q(\mu+1)}+\frac{\zeta+1}{q},$$

and so the value of this factor is limited. Therefore for any given positive number ϵ, we can determine a positive integer n_0, so that for all positive integral values of n, greater than n_0, we have the inequality

$$\left| a_{n,\mu} - \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{(np^{\mu} q)^{\nu}}{\nu !} \right| < \epsilon,$$

i.e.,

$$\left| a_{n,\mu} - \left(1-e^{-np^{\mu} q}\right) \right| < \epsilon.$$

College of Science,
Tohoku Imperial University,
Sendai, Japan.
April, 1928.

GEODESIC CURVES IN SOME TRIPLE REGIONS WITHIN FOUR-DIMENSIONAL FLAT SPACE

BY

A. R. FORSYTH

[*Read July 1, 1928*]

1. The investigations, which follow, are devoted mainly to the specific determination of geodesic curves in a few triple quadric regions that lie within flat space of four dimensions. To specify any surface in the three-dimensional Euclidean space, two parametric variables are necessary; and on any such surface, geodesic curves can be determined from a single ordinary differential equation of the second order. When we pass to abstract spaces of a greater number of dimensions and to geodesic curves in sub-spaces of more than two dimensions, the curves can be determined from a set of ordinary differential equations, each of the second order.

Every configuration in any region can be regarded as a configuration in an uncurved space of some increased number of dimensions. For the purpose of illustration, triple regions in a four-dimensional space are considered here: multiplicity of the region and of the enclosing space can be increased by a formal extension of the analysis.

Some preliminary definitions are stated. A space of one dimension is called a *line* when all the curvatures are zero: otherwise, it is called a *curve*. A space of two dimensions is called a *plane* when it is doubly resoluble into lines of which a single infinitude passes through each point; otherwise, it is called a *surface*. A space of three dimensions is called a *flat* when it is triply resoluble into lines of which a double infinitude passes through each point;* otherwise, it is called a *region*. In a space of m dimensions, sub-spaces of dimensions higher than m-1 do not exist: so that, in

* An uncurved space of m dimensions is called *homaloidal* by Clifford. It is m-ply resoluble into lines of which a $(p-1)$ple infinitude passes through each point.

four-dimensional space, the sub-spaces are a curve, a surface, and a region (including a line, a plane, a flat, respectively).

What follows is the analysis connected with geodesic curves in triple regions that are either ovoidal, or paraboloidal, or cyclo-ovoidal. with brief statements as to geodesic curves in dicyclic, sphero-quadric, and globular regions.

2. In the four-dimensional space, the co-ordinates of a point are taken to be x, y, z, v. In order to select a triple region, these co-ordinates are taken to be functions of three independent parameters p, q, r : and, for generality, it is assumed that the functions are independent of one another.

We write

$$\frac{\partial x}{\partial p}=x_1\ ;\ \frac{\partial x}{\partial q}=x_2,\ \ \frac{\partial x}{\partial r}=x_3\ ;\ \frac{\partial^2 x}{\partial p^2}=x_{11},\ \frac{\partial^2 x}{\partial p\,\partial q}=x_{12},\ldots,$$

and so for derivatives of y, z, and v Then an element ds of arc in the triple region is given by

$$ds^2=\mathrm{A}\,dp^2+\mathrm{B}\,dq^2+\mathrm{C}\,dr^2+2\mathrm{F}\,dq\,dr+2\mathrm{G}\,drdp+2\mathrm{H}\,dpdq$$

$$=(\mathrm{A, B, C, F, G, H}\between dp, dq, dr)^2,$$

where

$$\mathrm{A}=\Sigma x_1^2\quad,\quad \mathrm{B}=\Sigma x_2^2\quad,\quad \mathrm{C}=\Sigma x_3^2.$$

$$\mathrm{F}=\Sigma x_2x_3\quad,\quad \mathrm{G}=\Sigma x_3x_1\quad,\quad \mathrm{H}=\Sigma x_1x_2,$$

the summation in each quantity being for the derivatives of x, y, z, v. Also, we introduce eighteen quantities Γ_{lm}, Δ_{lm}, Θ_{lm}, (for $l,m=1, 2, 3$), according to the definitions*

$$\left.\begin{aligned}\mathrm{A}\Gamma_{11}+\mathrm{H}\Delta_{11}+\mathrm{G}\Theta_{11}&=\tfrac{1}{2}\mathrm{A}_1\\ \mathrm{H}\Gamma_{11}+\mathrm{B}\Delta_{11}+\mathrm{F}\Theta_{11}&=\mathrm{H}_1-\tfrac{1}{2}\mathrm{A}_2\\ \mathrm{G}\Gamma_{11}+\mathrm{F}\Delta_{11}+\mathrm{C}\Theta_{11}&=\mathrm{G}_1-\tfrac{1}{2}\mathrm{A}_3\end{aligned}\right\},$$

$$\left.\begin{aligned}\mathrm{A}\Gamma_{22}+\mathrm{H}\Delta_{22}+\mathrm{G}\Theta_{22}&=\mathrm{H}_2-\tfrac{1}{2}\mathrm{B}_1\\ \mathrm{H}\Gamma_{22}+\mathrm{B}\Delta_{22}+\mathrm{F}\Theta_{22}&=\tfrac{1}{2}\mathrm{B}_2\\ \mathrm{G}\Gamma_{22}+\mathrm{F}\Delta_{22}+\mathrm{C}\Theta_{22}&=\mathrm{F}_2-\tfrac{1}{2}\mathrm{B}_3\end{aligned}\right\},$$

* The quantities Γ, Δ, Θ, on the left hand sides are the Riemann Symbols $\left\{\begin{smallmatrix} ij \\ k\end{smallmatrix}\right\}$

$$\left.\begin{aligned} A\Gamma_{33}+H\Delta_{33}+G\Theta_{33}&=G_3-\tfrac{1}{2}C_1\\ H\Gamma_{33}+B\Delta_{33}+F\Theta_{33}&=F_3-\tfrac{1}{2}C_2\\ G\Gamma_{33}+F\Delta_{33}+C\Theta_{33}&=\tfrac{1}{2}C_3 \end{aligned}\right\},$$

$$\left.\begin{aligned} A\Gamma_{23}+H\Delta_{23}+G\Theta_{23}&=\tfrac{1}{2}(-F_1+G_2+H_3)\\ H\Gamma_{23}+B\Delta_{23}+G\Theta_{23}&=\tfrac{1}{2}B_3\\ G\Gamma_{23}+F\Delta_{23}+C\Theta_{23}&=\tfrac{1}{2}C_2 \end{aligned}\right\},$$

$$\left.\begin{aligned} A\Gamma_{31}+B\Delta_{31}+G\Theta_{31}&=\tfrac{1}{2}A_3\\ H\Gamma_{31}+B\Delta_{31}+F\Theta_{31}&=\tfrac{1}{2}(F_1-G_2+H_3)\\ G\Gamma_{31}+F\Delta_{31}+C\Theta_{31}&=\tfrac{1}{2}C_1 \end{aligned}\right\},$$

$$\left.\begin{aligned} A\Gamma_{12}+H\Delta_{12}+G\Theta_{12}&=\tfrac{1}{2}A_2\\ H\Gamma_{12}+B\Delta_{12}+F\Theta_{12}&=\tfrac{1}{2}B_1\\ G\Gamma_{12}+F\Delta_{12}+C\Theta_{12}&=\tfrac{1}{2}(F_1+G_2-H_3) \end{aligned}\right\}$$

Then the equations of a geodesic are

$$\left.\begin{aligned} p''+(\Gamma_{11},\ \Gamma_{22},\ \Gamma_{33},\ \Gamma_{23},\ \Gamma_{31},\ \Gamma_{12} \between p',\ q',r')^2&=0\\ q''+(\Delta_{11},\ \Delta_{22},\ \Delta_{33},\ \Delta_{23},\ \Delta_{31},\ \Delta_{12} \between p',q',\ r')^2&=0\\ r''+(\Theta_{11},\ \Theta_{22},\ \Theta_{33},\ \Theta_{23},\ \Theta_{31},\ \Theta_{12} \between p',q'.\ r')^2&=0 \end{aligned}\right\},$$

where $p'=\frac{dp}{ds}$, $p''=\frac{d^2p}{ds^2}$, and likewise for q and r; and these three equations are equivalent to two only, because

$$(A\ ,B,\ C,\ F,\ G,\ H \between p',\ q',\ r')^2=1.$$

When $F=0$, $G=0$, $H=0$, these equations become

$$\left.\begin{aligned} -2AP''&=\ \ A_1p'^2+2A_2p'q'+2A_3p'r'-B_1q'^2 \qquad\qquad -C_1r'^2\\ -2Bq''&=-A_2p'^2+2B_1p'q' \qquad\qquad +B_2q'^2+2B_3q'r'-C_2r'^2\\ -2Cr''&=-A_3p'^2 \qquad\qquad +2C_1p'r'-B_3q'^2+2C_2q'r^2+C_3r'^2 \end{aligned}\right\}.$$

as may be verified directly by minimising the integral

$$\sqrt{\left\{ A\left(\frac{dp}{dt}\right)^2 + B\left(\frac{dq}{dt}\right)^2 + C\left(\frac{dr}{dt}\right)^2 \right\}^{\frac{1}{2}}} \, dt.$$

Ovoidal Regions.

3. We take, as the fundamental definition, the equation

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + \frac{v^2}{d} = 1,$$

where the four quantities a, b, c, d, will be supposed to be different from one another (positive, though this assumption may be avoided), and ranged in decreasing order of magnitude. Using the extension of the idea of confocal quadrics, let p, q, r, be the roots (other than zero) of the equation

$$\frac{x^2}{a+w} + \frac{y^2}{b+w} + \frac{z^2}{c+w} + \frac{v^2}{d+w} = 1,$$

so that we have

$$-d > p > -c > q > -b > r > -a.$$

Then, if we write

$$f(\theta) = (\theta+a)(\theta+b)(\theta+c)(\theta+d),$$

we have

$$\left.\begin{aligned}
x^2 &= -\frac{a}{f'(-a)}(a+p)(a+q)(a+r) \\
y^2 &= -\frac{b}{f'(-b)}(b+p)(b+q)(b+r) \\
z^2 &= -\frac{c}{f'(-c)}(c+p)(c+q)(c+r) \\
v^2 &= -\frac{d}{f'(-d)}(d+p)(d+q)(d+r)
\end{aligned}\right\}.$$

By direct substitution, we find

$$\left.\begin{aligned} A &= \tfrac{1}{4}\frac{p(p-q)(p-r)}{(a+p)(b+p)(c+p)(d+p)} \\ B &= \tfrac{1}{4}\frac{q(q-p)(q-r)}{(a+q)(b+q)(c+q)(d+q)} \\ C &= \tfrac{1}{4}\frac{r(r-p)(r-q)}{(a+r)(b+r)(c+r)(d+r)} \\ F &= 0,\ G=0,\ H=0 \end{aligned}\right\}.$$

It is convenient to write

$$P=\tfrac{1}{4}\,\frac{p}{(a+p)(b+p)(c+p)(d+p)},$$

$$Q=\tfrac{1}{4}\,\frac{q}{(a+q)(b+q)(c+q)(d+q)},$$

$$R=\tfrac{1}{4}\,\frac{r}{(a+r)(b+r)(c+r)(d+r)};$$

and then

$$A=P(p-q)(p-r),\quad B=Q(q-p)(q-r),\quad C=R(r-p)(r-q).$$

4. The geodesic curves in the region are given by the second (and simpler) set of equations in § 2. As

$$\frac{A_1}{A}=\frac{P_1}{P}+\frac{1}{p-q}+\frac{1}{p-r},\qquad \frac{A_2}{A}=\frac{1}{q-p},\qquad \frac{A_3}{A}=\frac{1}{r-p},$$

$$-\frac{B_1}{A}=\frac{q-r}{(p-q)(p-r)}\,\frac{Q}{P},\quad -\frac{C_1}{A}=\frac{r-q}{(p-r)(p-q)}\,\frac{R}{P},$$

the first of the set of equations is

$$-2p''=\left(\frac{P_1}{P}+\frac{1}{p-q}+\frac{1}{p-r}\right)p'^2+2\,\frac{p'q'}{q-p}+2\,\frac{p'r'}{r-p}$$

$$+\frac{q-r}{(p-q)(p-r)}\,\frac{Q}{P}\,q'^2+\frac{r-q}{(p-r)(p-q)}\,\frac{R}{P}\,r'^2.$$

To obtain integrals of the equations, we introduce three quantities ξ, η, ζ, functions only of p, q, r, and not involving p', q', r', by the definitions

$$\frac{Pp'^2}{\xi}=\frac{Qq'^2}{\eta}=\frac{Rr'^2}{\zeta}.$$

Then the foregoing equation becomes

$$-\frac{2p''}{p'}-\frac{P_1}{P}p'=\frac{p'-2q'}{p-q}+\frac{p'-2r'}{p-r}-p'\left(\frac{1}{p-r}-\frac{1}{p-q}\right)\frac{\eta-\zeta}{\xi};$$

and the other two equations become

$$-\frac{2q''}{q'}-\frac{Q_2}{Q}q'=\frac{q'-2r'}{q-r}+\frac{q'-2p'}{q-p}-q'\left(\frac{1}{q-p}-\frac{1}{q-r}\right)\frac{\zeta-\xi}{\eta},$$

$$-\frac{2r''}{r'}-\frac{R_3}{R}r'=\frac{r'-2p'}{r-p}+\frac{r'-2q'}{r-q}-r'\left(\frac{1}{r-q}-\frac{1}{r-p}\right)\frac{\xi-\eta}{\zeta}.$$

From the equations defining ξ, η, ζ, we have

$$\frac{\xi'}{\xi}-\frac{\eta'}{\eta}=\frac{2p''}{p'}+\frac{P_1}{P}p'-\frac{2q''}{q'}-\frac{Q_2}{Q}q'$$

$$=-\frac{p'-2r'}{p-r}+\frac{q'-2r'}{q-r}+\frac{p'+q'}{p-q}$$

$$-p'\left(\frac{1}{r-p}-\frac{1}{q-p}\right)\frac{\eta-\zeta}{\xi}-q'\left(\frac{1}{q-p}-\frac{1}{q-r}\right)\frac{\zeta-\xi}{\eta},$$

with similar expressions for $\frac{\eta'}{\eta}-\frac{\zeta'}{\zeta}$, and $\frac{\zeta'}{\zeta}-\frac{\xi'}{\xi}$.

Now ξ, η, ζ, do not involve p', q', or r', and are functions of p, q, r: so that

$$\frac{\xi'}{\xi}=\frac{1}{\xi}\frac{\partial\xi}{\partial p}p'+\frac{1}{\xi}\frac{\partial\xi}{\partial q}q'+\frac{1}{\xi}\frac{\partial\xi}{\partial r}r',$$

and similarly for the others. When these are substituted, the equations are homogeneous and linear in p', q', r'; and all three equations will be satisfied if, simultaneously in each, the co-efficients of p' q', r', are respectively equal.

Comparing the co-efficients of r' in the expression for $\frac{\xi'}{\xi}-\frac{\eta'}{\eta}$, we have

$$\frac{1}{\xi}\frac{\partial \xi}{\partial r}-\frac{1}{\eta}\frac{\partial \eta}{\partial r}=\frac{2}{p-r}-\frac{2}{q-r},$$

and therefore

$$\frac{\xi(q-r)^{-2}}{\eta(p-r)^{-2}} = \text{a function of } p \text{ and } q \text{ only}=\theta(p, q).$$

Similarly, from the co-efficients of p' in the expression for $\frac{\eta'}{\eta}-\frac{\zeta'}{\zeta}$

$$\frac{\eta(r-p)^{-2}}{\zeta(q-p)^{-2}} = \text{a function of } q \text{ and } r \text{ only } =\phi(q, r);$$

and from the co-efficients of q' in the expression for $\frac{\zeta'}{\zeta}-\frac{\xi'}{\xi}$,

$$\frac{\zeta(q-p)^{-2}}{\xi(r-q)^{-2}} = \text{a function of } r \text{ and } p \text{ only}=\psi(r, p).$$

Hence $$\theta(p, q)\phi(q, r)\psi(r, p)=1,$$

and so

$$\theta(p, q)=\frac{f(p)}{g(q)}, \qquad \phi(q, r)=\frac{g(q)}{h(r)}, \qquad \psi(r, p)=\frac{h(r)}{f(p)}.$$

As we are concerned only with the ratios of ξ, η, ξ, all the requirements are met by taking

$$\xi=(q-r)^2 f(p), \eta=(r-p)^2 g(q), \xi=(p-q)^2 h(r),$$

where $f(p)$, $g(q)$, $h(r)$, are functions of their respective single arguments to be determined.

Again, comparing the co-efficients of p' in the expression for

$$\frac{\xi'}{\eta'}-\frac{\eta'}{\eta},$$

we have

$$\frac{1}{\xi}\frac{\partial\xi}{\partial p}-\frac{1}{\eta}\frac{\partial\eta}{\partial p}=\frac{1}{r-p}+\frac{1}{p-q}-\left(\frac{1}{r-p}-\frac{1}{q-p}\right)\frac{\eta-\zeta}{\xi},$$

or, on substituting the newly assigned values of ξ, η, ζ,

$$\frac{f'(p)}{f(p)}=\frac{1}{p-r}+\frac{1}{p-q}-\left(\frac{1}{r-p}-\frac{1}{q-p}\right)\frac{\eta-\zeta}{\xi};$$

and therefore

$$(p-r)(p-q)f'(p)=(2p-q-r)f(p)-\frac{1}{q-r}\{(p-r)^2g(q)-(p-q)^2h(r)\}.$$

Differentiate this relation with respect to p: then

$$(p-r)(p-q)f''(p)=2f(p)-\frac{2}{q-r}\{(p-r)g(q)-(p-q)h(r)\},$$

that is,

$$-f''(p)(p-q)(q-r)(r-p)=2\{(q-r)f(p)+(r-p)g(q)+(p-q)h(r)\}.$$

Similarly, we find the same value for

$$-g''(q)(p-q)(q-r)(r-p),-h''(r)(p-q)(q-r)(r-p),$$

and therefore

$$f''(p)=g''(q)=h''(r),$$

so that each of them is a constant, say 2γ. Thus

$$f(p)=\gamma p^2+2\alpha p+\beta,\ g(q)=\gamma q^2+2\alpha' q+\beta',\ h(r)=\gamma r^2+2\alpha'' r+\beta''.$$

When these values are substituted in any one of the relations giving $f''(p)$, $g''(q)$, $h''(r)$, the relation

$$(q-r)(2\alpha p+\beta)+(r-p)(2\alpha' q+\beta')+(p-q)(2\alpha'' r+\beta'')=0$$

must be satisfied; and therefore

$$\alpha=\alpha'=\alpha'',\ \beta=\beta'=\beta''.$$

Moreover, γ can manifestly be made unity without loss of generality; and then α and β are two arbitrary constants of integration, determinable by the arbitrary initial direction of the geodesic curve. Thus

$$\xi=(q-r)^2 f(p)=(q-r)^2(p+u)(p+v),$$

$$\eta=(r-p)^2 g(q)=(r-p)^2(q+u)(q+v),$$

$$\zeta=(p-q)^2 h(r)=(p-q)^2(r+u)(r+v),$$

where now u and v are the arbitrary constants of integration.

5. To express the integrated equations of geodesics, write

$$S(\theta)=\theta(a+\theta)(b+\theta)(c+\theta)(d+\theta)(u+\theta)(v+\theta),$$

for $\theta=p, q, r$. Then

$$\frac{Pp'^2}{\xi}=\frac{p^2p'^2}{(q-r)^2S(p)}, \quad \frac{Qq'^2}{\eta}=\frac{q^2q'^2}{(r-p)^2S(q)}, \quad \frac{Rr'^2}{\zeta}=\frac{r^2r'^2}{(p-q)^2S(r)};$$

and these quantities are equal to one another, say $=\Omega^2$. Thus

$$\frac{p'}{\{S(p)\}^{\frac{1}{2}}}=\frac{q-r}{p}\,\Omega,$$

$$\frac{q'}{\{S(q)\}^{\frac{1}{2}}}=\frac{r-p}{q}\,\Omega,$$

$$\frac{r'}{\{S(r)\}^{\frac{1}{2}}}=\frac{p-q}{r}\,\Omega;$$

and so we have

$$\left.\begin{aligned}\frac{p\,dp}{2\{S(p)\}^{\frac{1}{2}}}+\frac{q\,dq}{2\{S(q)\}^{\frac{1}{2}}}+\frac{r\,dr}{2\{S(r)\}^{\frac{1}{2}}}=0\\ \frac{p^2dp}{2\{S(p)\}^{\frac{1}{2}}}+\frac{q^2dq}{2\{S(q)\}^{\frac{1}{2}}}+\frac{r^2dr}{2\{S(r)\}^{\frac{1}{2}}}=0\end{aligned}\right\}.$$

Further,

$$4ds^2 = 4(\mathrm{A}dp^2 + \mathrm{B}dq^2 + \mathrm{C}dr^2)$$

$$= \Sigma \frac{p(p-q)(p-r)dp^2}{(a+p)(b+p)(c+p)(d+p)}$$

$$= -\Omega^2(q-r)(r-p)(p-q)ds^2 \Sigma(u+p)(v+p)(q-r)$$

$$= \Omega^2(q-r)^2(r-p)^2(p-q)^2 ds^2,$$

or

$$\Omega = \frac{2}{(q-r)(r-p)(p-q)}.$$

Finally

$$\Sigma \frac{p^3 p'}{2\{\mathrm{S}(p)\}^{\frac{1}{2}}} = \frac{1}{2}\, \Sigma p^2(q-r)\Omega = -1,$$

so that

$$\frac{p^3 dp}{2\{\mathrm{S}(p)\}^{\frac{1}{2}}} + \frac{q^3 dq}{2\{\mathrm{S}(q)\}^{\frac{1}{2}}} + \frac{r^3 dr}{2\{\mathrm{S}(r)\}^{\frac{1}{2}}} = -ds.$$

Also it is easy to see that the relation

$$\frac{dp}{2\{\mathrm{S}(p)\}^{\frac{1}{2}}} + \frac{dq}{2\{\mathrm{S}(q)\}^{\frac{1}{2}}} + \frac{dr}{2\{\mathrm{S}(r)\}^{\frac{1}{2}}} = -\frac{ds}{pqr}$$

is satisfied.

It thus appears that, for the expression of the geodesic curves, we require the limited Abelian functions of genus three discussed by Weierstrass.*

In order to secure reality for the geodesics, the quantities $\mathrm{S}(p)$, $\mathrm{S}(q)$, $\mathrm{S}(r)$, must be positive: hence, with the assumptions made concerming a, b, c, d, we must have

$$(u+p)(v+p) > 0,\ (u+q)(v+q) < 0,\ (u+r)(v+r) > 0,$$

so that

$$p > -u > q > -v > r.$$

thus u and c lie between $-p$ and $-q$; we can have three cases, $u > c$, $u = c$, $u < c$. Also u and b lie between $-q$ and $-r$; again we can have three cases, $v > b$, $v = b$, $v < b$. Thus there are nine cases to be considered in the delineation of the geodesic curves.

* *Crelle*, Vol. LII (1856), pp. 285-339; *Ges. Werke*, Vol. I, pp. 297-355.

Paraboloidal Regions.

6. We take, as the fundamental definition, the equation

$$4x=\frac{y^2}{a}+\frac{z^2}{b}+\frac{v^2}{c},$$

where the three quantities a, b, c, are supposed to be unequal to one another, positive, and arranged in descending order of magnitude.

To obtain parametric expressions for the co-ordinates of any point in the region, we again use the extended idea of confocal paraboloids. Let p, q, r, be the roots (other than zero) of the equation

$$\frac{y^2}{a+\mu}+\frac{z^2}{b+\mu}+\frac{v^2}{c+\mu}=4(x+\mu);$$

the roots are unequal, and we have

$$-c>p>-b>q>-a>r.$$

Then

$$\left.\begin{aligned}
y^2&=-\frac{4a}{(a-b)(a-c)}(a+p)(a+q)(a+r)\\
z^2&=-\frac{4b}{(b-a)(b-c)}(b+p)(b+q)(b+r)\\
v^2&=-\frac{4r}{(c-a)(c-b)}(c+p)(c+q)(c+r)\\
x&=-a-b-c-p-q-r
\end{aligned}\right\}.$$

By direct substitution, we find

$$\left.\begin{aligned}
A&=\frac{p(p-q)(p-r)}{(p+a)(p+b)(p+c)}=(p-q)(p-r)P\\
B&=\frac{q(q-r)(q-p)}{(q+a)(q+b)(q+c)}=(q-r)(q-p)Q\\
C&=\frac{r(r-p)(r-q)}{(r+a)(r+b)(r+c)}=(r-p)(r-q)R\\
F&=0,\ G=0,\ H=0
\end{aligned}\right\}$$

Except for the changed forms of P, Q, R, the early stage of the analysis for the determination of geodesic curves is exactly the same as the analysis in §4. We write

$$T(\theta)=\theta(a+\theta)(b+\theta)(c+\theta)(u+\theta)(v+\theta),$$

where u and v are arbitrary constants; and we have the integrals of the equations in the form

$$\frac{p^2p'^2}{(q-r)^2T(p)}=\frac{q^2q'^2}{(r-p)^2T(q)}=\frac{r^2r'^2}{(p-q)^2T(r)}=\Phi^2.$$

Thus

$$\left.\begin{aligned}\frac{p\,dp}{2\{T(p)\}^{\frac{1}{2}}}+\frac{q\,dq}{2\{T(q)\}^{\frac{1}{2}}}+\frac{r\,dr}{2\{T(r)\}^{\frac{1}{2}}}=0\\ \frac{p^2dp}{2\{T(p)\}^{\frac{1}{2}}}+\frac{q^2dq}{2\{T(q)\}^{\frac{1}{2}}}+\frac{r^2dr}{2\{T(r)\}^{\frac{1}{2}}}=0\end{aligned}\right\}.$$

As before for Ω, so here for Φ, we find

$$\Phi=\frac{1}{(p-q)(q-r)(r-p)}.$$

Also

$$\frac{p^3p'}{\{T(p)\}^{\frac{1}{2}}}+\frac{q^3q'}{\{T(q)\}^{\frac{1}{2}}}+\frac{r^3r'}{\{T(r)\}^{\frac{1}{2}}}=\{p^2(q-r)+q^2(r-p)+r^2(p-q)\}\Phi=-1,$$

so that

$$\frac{p^3dp}{\{T(p)\}^{\frac{1}{2}}}+\frac{q^3dq}{\{T(q)\}^{\frac{1}{2}}}+\frac{r^3dq}{\{T(r)\}^{\frac{1}{2}}}=-ds;$$

and also

$$\frac{dp}{\{T(p)\}^{\frac{1}{2}}}+\frac{dq}{\{T(q)\}^{\frac{1}{2}}}+\frac{dr}{\{T(r)\}^{\frac{1}{2}}}=-\frac{ds}{pqr}.$$

Hence for the expression of the geodesic curves in paraboloidal triple regions in space of four dimensions, the customary hyperelliptic functions of genus two suffice. Further, for reality, u lies between $-p$ and $-q$; and v lies between $-q$ and $-r$; so that, for the complete delineation of the geodesic curves, there are nine cases to be considered.

Cyclo-ovoidal Regions.

7. As the fundamental definition, we take the equation

$$\frac{x^2+y^2}{a}+\frac{z^2}{b}+\frac{v^2}{c}=1,$$

with the sole restrictions that a, b, c, are unequal and that not all of these are negative. For the parametric expression of the co-ordinates, we write *

$$x=\rho\cos\theta,\ y=\rho\sin\theta\ ;$$

and then we have

$$\frac{\rho^2}{a}+\frac{z^2}{b}+\frac{v^2}{c}=1.$$

With the customary ellipsoidal co-ordinates in three dimensions, we can take

$$A=\rho^2=\frac{a(a+p)(a+q)}{(a-b)(a-c)},$$

$$z^2=\frac{b(b+p)(b+q)}{(b-c)(b-a)},$$

$$v^2=\frac{c(c+p)(c+q)}{(c-a)(c-b)}.$$

* As the region is one of revolution round the planes $z=0$, $v=0$, one equation of geodesic curves is

$$\frac{1}{x}\frac{d^2x}{ds^2}=\frac{1}{y}\frac{d^2y}{ds^2},$$

deduced from the property that the prime radius of curvature of the geodesic curve coincides with the normal to the region. Thus one integral of the equations is

$$\rho^2\frac{d\theta}{ds}=\text{constant}\ :$$

in the text, it will be obtained in the regular course of the analysis.

By direct substitution it follows, as usual, that

$$d\rho^2 + dz^2 + dv^2 = \mathrm{B}dp^2 + \mathrm{C}dq^2,$$

where

$$\mathrm{B} = \tfrac{1}{4}\frac{p(p-q)}{(a+p)(b+p)(c+p)} = (p-q)\mathrm{P},$$

$$\mathrm{C} = \tfrac{1}{4}\frac{q(q-p)}{(a+q)(b+q)(c+q)} = (q-p)\mathrm{Q}.$$

Now the element ds of arc in the region is given by

$$ds^2 = \rho^2 d\theta^2 + d\rho^2 + dz^2 + dv^2$$

$$= \mathrm{A}d\theta^2 + \mathrm{B}dp^2 + \mathrm{C}dq^2,$$

with the foregoing values of A, B, C: and it is to be noted that

$$\mathrm{F}=0, \quad \mathrm{G}=0, \quad \mathrm{H}=0,$$

while A, B, C, do not explicitly involve the parametric variable θ, a property which simplifies the form of the equations of the geodesics.

For any quantity Θ, let

$$\frac{\partial\Theta}{\partial\theta} = \Theta_1, \quad \frac{\partial\Theta}{\partial p} = \Theta_2, \quad \frac{\partial\Theta}{\partial q} = \Theta_3:$$

then the equations of geodesic curves are

$$\left.\begin{aligned}
-2\mathrm{A}\theta'' &= 2\mathrm{A}_2\theta' p' + 2\mathrm{A}_3\theta' q' \\
-2\mathrm{B}p'' &= -\mathrm{A}_2\theta'^2 + \mathrm{B}_2 p'^2 + 2\mathrm{B}_3 p'q' - \mathrm{C}_2 q'^2 \\
-2\mathrm{C}q'' &= -\mathrm{A}_3\theta'^2 - \mathrm{B}_3 p'^2 + 2\mathrm{C}_2 p'q' + \mathrm{C}_3 q'^2
\end{aligned}\right\},$$

the absence of the other terms being due to zero co-efficients.

8. From the first of these equations, we have

$$\frac{\theta''}{\theta'}+\frac{A_2 p'+A_3 q'}{A}=0,$$

the integral of which is

$$A\theta'=\text{constant}=u,$$

and is the integral indicated in the preceding foot-note.

To complete the integration, we introduce two new variables t and ω, defined by the equations

$$\frac{Pp'^2}{t+p}=\frac{Qq'^2}{t+q}=\frac{A\theta'^2}{\omega}.$$

Then, as $B=(p-q)P$, $C=(q-p)Q$, and

$$A\theta'^2+Bp'^2+Cq'^2=1, \quad A\theta'=u,$$

we have, on substitution

$$\omega=(p-q)^2\frac{u^2}{A-u^2}.$$

Let

$$u^2=\frac{a}{(a-b)(a-c)}k^2, \qquad A=\frac{a}{(a-b)(a-c)}\alpha,$$

so that k is a modified constant in place of u, and

$$\alpha=(a+p)(a+q),$$

while the postulated equations are

$$\frac{Pp'^2}{t+p}=\frac{Qq'^2}{t+q}=\frac{\alpha-k^2}{\alpha}\quad\frac{1}{(p-q)^2}.$$

Taking logarithmic derivatives of the equal first and third fractions, we have

$$\frac{2p''}{p'}+\frac{P_1}{P}p'-\frac{t'+p'}{t+p}=\frac{k^2a'}{a(a-k^2)}-2\frac{p'-q'}{p-q}.$$

From the second of the equations of geodesics, we have

$$-2p''=-\frac{A_2}{B}\theta'^2+\frac{B_2}{B}p'^2+2\frac{B_3}{B}p'q'-\frac{C_2}{B}q'^2$$

$$=-\frac{k^2a_2}{a(a-k^2)}\frac{p-q}{t+p}p'^2+\frac{B_2}{B}p'^2+2\frac{B_3}{B}p'q'+\frac{C_2}{C}\frac{t+q}{t+p}p'^2,$$

and therefore

$$-2\frac{p''}{p'}=-\frac{k^2}{a-k^2}\frac{p-q}{(t+p)(a+p)}p'+\left(\frac{1}{p-q}+\frac{P_1}{P}\right)p'$$

$$+\frac{2q'}{q-p}+\frac{1}{p-q}\frac{t+q}{t+p}p',$$

that is,

$$2\frac{p''}{p}+\frac{P_1}{P}p'-\frac{t'+p'}{t+p}=-\frac{t'}{t+p}-2\frac{p'-q'}{p-q}+\frac{k^2}{a-k^2}\frac{p-q}{(t+p)(a+p)}p'.$$

Comparing the two equations, we have

$$\frac{k^2a'}{a(a-k^2)}=-\frac{t'}{t+p}+\frac{k^2}{a-k^2}\frac{p-q}{(t+p)(a+p)}p'.$$

Similarly treating the third equation of geodesics, we find

$$\frac{k^2a'}{a(a-k^2)}=-\frac{t'}{t+q}+\frac{k^2}{a-k^2}\frac{q-p}{(t+q)(a+q)}q'.$$

As t is a function of p and q, these equations are satisfied by equating co-efficients of p' on the two sides of each equation and likewise for the co-efficients of q'. This happens for the first equation if

$$-\frac{1}{t+p}\frac{\partial t}{\partial q}=\frac{k^2}{a-k^2}\frac{1}{a+q};$$

$$-\frac{1}{t+p}\frac{\partial t}{\partial p}=\frac{k^2}{a-k^2}\frac{1}{a+p}-\frac{k^2}{a-k^2}\frac{p-q}{(t+p)(a+p)}$$

$$=\frac{k^2}{a-k^2}\frac{t+q}{(t+p)(a+p)},$$

that is, the second relation is

$$-\frac{1}{t+q}\frac{\partial t}{\partial p}=\frac{k^2}{a-k^2}\frac{1}{a+p}.$$

The same two relations follow from the second equation; thus

$$\left.\begin{aligned}\frac{\partial t}{\partial p}+\frac{k^2}{a-k^2}\frac{t+q}{a+p}=0\\ \frac{\partial t}{\partial q}+\frac{k^2}{a-k^2}\frac{t+p}{a+q}=0\end{aligned}\right\}.$$

Moreover, these satisfy the necessary condition

$$\frac{\partial}{\partial q}\left(\frac{\partial t}{\partial p}\right)=\frac{\partial}{\partial p}\left(\frac{\partial t}{\partial q}\right),$$

when account of a is taken. Thus the equation for t is

$$dt+\frac{k^2}{a-k^2}\left(\frac{t+q}{a+p}dp+\frac{t+p}{a+p}dq\right)=0;$$

and its integral is

$$t=\frac{k^2}{a-k^2}(p+q)+\frac{ak^2}{a-k^2}+\text{A}\frac{a}{a-k^2},$$

where A is an arbitrary constant. As k^2 also is an arbitrary constant, we have the elements necessary for the primitive of the geodesic equations, which is derivable as follows. We have

$$
\begin{aligned}
(t+p)(a-k^2) &= k^2(p+q)+ak^2+\mathrm{A}a+pa-pk^2 \\
&= \mathrm{A}a+pa+k^2(a+q) \\
&= a\left\{ \mathrm{A}+p+\frac{k^2}{a+p} \right\} \\
&= \frac{a}{a+p}(p+\rho)(p+\sigma),
\end{aligned}
$$

where ρ and σ are arbitrary constants taking the place of A and k^2 according to the relation

$$
\begin{aligned}
\mathrm{A} &= \rho+\sigma-a, \\
k^2 &= (a-\rho)(a-\sigma),
\end{aligned}
$$

and so

$$u^2 = a\frac{(a-\rho)(a-\sigma)}{(a-b)(a-c)}.$$

Now

$$\frac{\mathrm{P}p'^2}{t+p} = \frac{a-k^2}{a}\,\frac{1}{(p-q)^2};$$

and therefore

$$\mathrm{P}p'^2(p-q)^2 = (t+p)\,\frac{a-k^2}{a} = \frac{(p+\rho)(p+\sigma)}{a+p}:$$

that is,

$$\frac{1}{4}\,\frac{p(p-q)^2}{(b+p)(c+p)}\,p'^2 = (p+\rho)(p+\sigma)$$

Write

$$f(\theta) = \theta(\theta+b)(\theta+c)(\theta+\rho)(\theta+\sigma);$$

then

$$\frac{1}{2}\,p(p-q)\,\frac{dp}{ds} = \{f(p)\}^{\frac{1}{2}}.$$

Similarly

$$\frac{1}{2}\, q(q-p)\frac{dq}{ds}=\{f(q)\}^{\frac{1}{2}}.$$

Hence

$$\frac{pdp}{2\{f(p)\}^{\frac{1}{2}}}+\frac{qdq}{2\{f(q)\}^{\frac{1}{2}}}=0.$$

Take

$$\frac{dp}{2\{f(p)\}^{\frac{1}{2}}}+\frac{dq}{2\{f(q)\}^{\frac{1}{2}}}=dw\,;$$

then we easily have

$$ds=-pq\ dw.$$

These equations are similar to those of geodesics on ellipsoids in Euclidean triple space.*

Finally, for the angle θ, we have

$$\frac{a}{(a-b)(a-c)}\,(a+p)(a+q)d\theta=u\ ds,$$

that is,

$$a(a+p)(a+q)d\theta=-\{-f(a)\}^{\frac{1}{2}}pq\ dw.$$

We thus have the primitive of the equations of the geodesics. It manifestly involves the simplest class (of genus two) of hyperelliptic functions.

Dicyclic Regions.

9. As the fundamental definition, we take the equation

$$\frac{x^2+y^2}{a}+\frac{z^2+v}{b}=1.$$

* See my *Lectures on Differential Geometry*, p. 147.

The co-ordinates of any point in the region can be expressed in the

$$x=\rho\cos\theta,\ y=\rho\sin\theta,\ z=\sigma\cos\phi,\ v=\sigma\sin\phi,$$

$$\rho^2=\frac{a}{a-b}(a+\lambda),\ \sigma^2=\frac{b}{b-a}(b+\lambda).$$

Then

$$ds^2=d\rho^2+\rho^2d\theta^2+d\sigma^2+\sigma^2d\phi^2$$

$$=\frac{a}{a-b}(a+\lambda)d\theta^2+\frac{b}{b-a}(b+\lambda)d\phi^2+\frac{1}{4}\ \frac{\lambda}{(a+\lambda)(b+\lambda)}$$

Either directly from the equations of the geodesics, or by using th of the equation arising out of revolution round the plane (x, y) and plane of (z, v), two integrals of the equations are

$$\rho^2\frac{d\theta}{ds}=\text{constant}=\alpha\, a^{\frac{1}{2}},$$

$$\sigma^2\frac{d\phi}{ds}=\text{constant}=\beta b^{\frac{1}{2}},$$

where α and β are (initial) arbitrary constants. Then

$$1=\alpha^2\frac{a-b}{a+\lambda}+\beta^2\frac{b-a}{b+\lambda}+\frac{1}{4}\frac{\lambda}{(a+\lambda)(b+\lambda)}\lambda'^2,$$

so that

$$\frac{1}{4}\lambda\lambda'^2=(a+\lambda)(b+\lambda)-(a-b)\{\alpha^2(b+\lambda)-\beta^2(a+\lambda)\}$$

$$=(\xi-\lambda)(\eta-\lambda),$$

where ξ and η can be regarded as two arbitrary constants, in place β, according to the relations

$$(a+\xi)(a+\eta)=\alpha'a-b),\ (b+\xi)(b+\eta)=\beta(b-a).$$

Thus the equations of the geodesic, expressing the parameters θ, ϕ, λ in association with the arc s, are

$$\left.\begin{aligned} ds&=\tfrac{1}{2}\left\{\frac{\lambda}{(\xi-\lambda)(\eta-\lambda)}\right\}^{\frac{1}{2}}d\lambda \\ \left\{\frac{a}{(a+\xi)(a+\eta)}\right\}^{\frac{1}{2}}d\theta&=\tfrac{1}{2}\left\{\frac{\lambda}{(\xi-\lambda)(\eta-\lambda)}\right\}^{\frac{1}{2}}\frac{d\lambda}{a+\lambda} \\ \left\{\frac{b}{(b+\xi)(b+\eta)}\right\}^{\frac{1}{2}}d\phi&=\tfrac{1}{2}\left\{\frac{\lambda}{(\xi-\lambda)(\eta-\lambda)}\right\}^{\frac{1}{2}}\frac{d\lambda}{b+\lambda}\end{aligned}\right\},$$

in which ξ and η are arbitrary constants.

The explicit expressions manifestly involve elliptic functions.

Sphero-quadric Regions.

10. As the fundamental equation, we take

$$\frac{x^2+y^2+z^2}{a}+\frac{v^2}{b}=1.$$

For the co-ordinates, x, y, z, v, we take

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta,$$

$$r^2=\frac{a}{a-b}(a+\lambda),\quad v^2=\frac{b}{b-a}(b+\lambda);$$

and then

$$ds^2=r^2d\theta^2+r^2\sin^2\theta\,d\phi^2+dr^2+dv^2$$

$$=\frac{a}{a-b}(a+\lambda)(d\theta^2+\sin^2\theta\,d\phi^2)+\tfrac{1}{4}\frac{\lambda}{(a+\lambda)(b+\lambda)}d\lambda^2.$$

We write $A=\frac{a}{a-b}(a+\lambda)$, $B=A\sin^2\theta$. Then the ϕ-equation of the geodesic curves is

$$-2B\phi''=2B_1\theta'\phi'+2B_3\phi'\lambda',$$

where $B_1 = \dfrac{\partial B}{\partial \theta}$, $B_3 = \dfrac{\partial B}{\partial \lambda}$; and its first integral is

$$B\phi' = \text{constant} = l.$$

The θ-equation of the geodesic curves is

$$-2A\theta'' = 2A_3\theta'\lambda' - B_1\phi'^2$$

$$= 2A_3\theta'\lambda' - \frac{2l^2}{B}\cot\theta,$$

that is,

$$\frac{d}{ds}(A\theta') = \frac{l^2}{A}\,\frac{\cos\theta}{\sin^3\theta},$$

and therefore an integral is

$$A^2\theta'^2 = \gamma^2 - \frac{l^2}{\sin^2\theta},$$

where γ is an arbitrary constant. The permanent equation

$$1 = A(\theta'^2 + \sin^2\theta\phi'^2) + \tfrac{1}{4}\,\frac{\lambda\lambda'^2}{(a+\lambda)(b+\lambda)}$$

now becomes

$$\lambda + c = \tfrac{1}{4}\,\frac{\lambda\lambda'^2}{b+\lambda},$$

where c is a new arbitrary constant in place of γ, such that

$$c = a - \frac{b-a}{a}\gamma^2.$$

Hence the equations are

$$\left.\begin{aligned} ds &= \tfrac{1}{2}\left\{\frac{\lambda}{(b+\lambda)(c+\lambda)}\right\}^{\frac{1}{2}} d\lambda \\ \frac{\sin\theta\, d\theta}{(\gamma^2\sin^2\theta - l^2)^{\frac{1}{2}}} &= \frac{a-b}{2a}\left\{\frac{\lambda}{(b+\lambda)(c+\lambda)}\right\}^{\frac{1}{2}} \frac{d\lambda}{a+\lambda} \\ \sin^2\theta\, d\phi &= \frac{l}{2a}(a-b)\left\{\frac{\lambda}{(b+\lambda)(c+\lambda)}\right\}^{\frac{1}{2}} \frac{d\lambda}{a+\lambda} \end{aligned}\right\},$$

manifestly demanding elliptic functions for a complete explicit expression.

Globular Regions.

11. The fundamental equation is taken in the form

$$x^2+y^2+z^2+v^2=1.$$

Either from the property that the direction of prime curvature in the geodesic coincides with the normal to the region at each point, or from the equations of the geodesics with assumptions of parametric expressions such as $x=\sin p \sin q$, $y=\sin p \cos q$, $z=\cos p \cos r$, $v=\cos p \sin r$, for the co-ordinates we can obtain the integral equations in the form

$$\left.\begin{aligned}
x&=\cos\alpha' \cos\gamma' \sin s-\sin\alpha' \cos\delta' \cos s\\
y&=\sin\alpha' \cos\gamma' \sin s+\cos\alpha' \cos\delta' \cos s\\
z&=\sin\beta' \sin\gamma' \sin s+\cos\beta' \sin\delta' \cos s\\
v&=\cos\beta' \sin\gamma' \sin s-\sin\beta' \sin\delta' \cos s
\end{aligned}\right\}.$$

The curves manifestly are equatorial circles : the equations of the plane of any geodesic are

$$\left\|\begin{matrix} x, & y, & z, & v\\ \cos\alpha' \cos\gamma', & \sin\alpha' \cos\gamma', & \sin\beta' \sin\gamma', & \cos\beta' \sin\gamma'\\ -\sin\alpha' \cos\delta', & \cos\alpha' \cos\delta', & \cos\beta' \sin\delta', & -\sin\beta' \sin\delta' \end{matrix}\right\|=0,$$

For, with the assumed values of x, y, z, v, we have

$$A=1,\ B=\sin^2 p,\ C=\cos^2 p,\ F=0,\ G=0,\ H=0.$$

The equation of the arc element is therefore

$$ds^2=dp^2+\sin^2 p\, dq^2+\cos^2 p\, dr^2.$$

Two integrals of the geodesic equations are

$$\sin^4 p.\, q'=\beta,\quad \cos^4 p.\, r'=\gamma,$$

where β and γ are constants. The derivation of the foregoing expressions of the co-ordinates depends upon only simple quadratures.

The integral equations can be expressed differently so as to involve the parameters p, q, r, without the intervention of the arc as the current variable. Because

$$p'^2 = 1 - q'^2 \sin^2 p - r'^2 \cos^2 p$$

$$= 1 - \frac{\beta^2}{\sin^2 p} - \frac{\gamma^2}{\cos^2 p},$$

we have

$$\left.\begin{aligned} \frac{dq}{dp} &= \beta \frac{\cos p}{\sin p} \left(\sin^2 p \cos^2 p - \beta^2 \cos^2 p - \gamma^2 \sin^2 p\right)^{-\frac{1}{2}} \\ \frac{dr}{dp} &= \gamma \frac{\sin p}{\cos p} \left(\sin^2 p \cos^2 p - \beta^2 \cos^2 p - \gamma^2 \sin^2 p\right)^{-\frac{1}{2}} \end{aligned}\right\},$$

which can be regarded as the parametric equations of the geodesics. These, when integrated, give

$$\left.\begin{aligned} t \tan (q+\lambda) &= \left(\frac{1-\beta t}{1-\frac{\beta}{t}}\right)^{\frac{1}{2}} \tan (\gamma+\mu) \\ &= \left(\frac{\sin^2 p - \beta t}{\frac{\beta}{t} - \sin^2 p}\right)^{\frac{1}{2}} \end{aligned}\right\},$$

where λ and μ are arbitrary constants of integration, and the constant t is a root of the quadratic

$$\frac{\beta}{t} + \beta t = 1 + \beta^2 - \gamma^2.$$

As the two roots of this quadratic have unity for their product, we may take t to be the smaller root; and then, for the range of p,

$$\frac{\beta}{t} > \sin^2 p > \beta t.$$

12. As a mere illustration of the analysis, take a point

$$p_0 = \tfrac{1}{4}\pi, \quad q_0 = \tfrac{1}{4}\pi, \quad r_0 = \tfrac{1}{4}\pi,$$

in the globular region and consider the cluster of geodesics through this point for which

$$q'_0=r'_0,$$

that is, for which

$$\beta=\gamma.$$

Then, as the equation for t now is

$$\frac{\beta}{t}+\beta t=1,$$

so that

$$\frac{1-\beta t}{1-\frac{\beta}{t}}=\frac{\frac{\beta}{t}}{t\beta}=\frac{1}{t^2},$$

the two equations for the cluster of geodesics are

$$t\tan(q+\lambda)=\frac{1}{t}\tan(r+\mu)=\left(\frac{\sin^2 p-\beta t}{\frac{\beta}{t}-\sin^2 p}\right)^{\frac{1}{2}}.$$

A simpler explicit form can be obtained. Let

$$\beta=\tfrac{1}{2}\cos\epsilon=\gamma,$$

so that

$$t=\sec\epsilon-\tan\epsilon.$$

To determine λ and μ, we have, at the initial point,

$$\frac{\sin^2 p-\beta t}{\frac{\beta}{t}-\sin^2 p}=\frac{\frac{1}{2}-\beta\epsilon}{\frac{\beta}{t}-\frac{1}{2}}=1\,;$$

thus

$$\tan(\tfrac{1}{4}\pi+\lambda)=\frac{1}{t}=\sec\epsilon+\tan\epsilon=\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\epsilon),$$

$$\tan(\tfrac{1}{4}\pi+\mu)=t=\sec\epsilon-\tan\epsilon=\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\epsilon),$$

and therefore

$$\lambda=\tfrac{1}{2}\epsilon,\qquad \mu=-\tfrac{1}{2}\epsilon.$$

The equation between q and p can be changed to

$$\frac{\cos 2p}{\sin \epsilon}=\frac{\sin \epsilon+\cos(2q+\epsilon)}{1+\sin \epsilon \cos (2q+\epsilon)},$$

that is,

$$\cos (2q+\epsilon) \sin \epsilon=\frac{\cos 2p-\sin^2\epsilon}{1-\cos 2p};$$

and the equation between r and p to

$$\cos (2r-\epsilon) \sin \epsilon=\frac{\cos 2p+\sin^2\epsilon}{1+\cos 2p},$$

which seem the simplest integrated forms.

Consequently the cluster of geodesics through the point $p=\frac{1}{4}\pi$, $q=\frac{1}{4}\pi$, $r=\frac{1}{4}\pi$ (that is, through the point $x=y=z=v=\frac{1}{2}$), such that $q'=r'$ at their initial common pole, all lie on the surface the equation of which results from the simple elimination of ϵ between the two foregoing relations.

It should however be noted that the surface, thus defined by the resulting equation between p, q, r, is not a "geodesic surface" in the sense required for Riemann's definition of the measure of curvature at any point of a triple region, estimated for any orientation.

10th May, 1928.

PRESIDENTIAL ADDRESS

BY

PROFESSOR GANESH PRASAD.

DEAR COLLEAGUES OF THE CALCUTTA MATHEMATICAL SOCIETY AND
HONOURED GUESTS,

We are assembled this afternoon to celebrate the founding of the Calcutta Mathematical Society in 1908 with the avowed object of promoting mathematical research and establishing at Calcutta a centre at which all desirous of extending the bounds of mathematical knowledge could meet and have the merits of their contributions adjudicated upon by an indigenous tribunal working independently of any institution in India or outside India. That the founding of such a Society was an event of no mean importance in the intellectual life of India in general and of Bengal in particular, is evidenced by the presence in this gathering not only of representative mathematicians from various parts of India but also of distinguished laymen who have rendered eminent services to the country in the sphere of education. In the name of the Calcutta Mathematical Society, I beg to offer to all the guests a most cordial welcome and to thank them for their presence at this function.

As the President at this meeting, I have the privilege of making remarks on such topics as may suit the occasion. I am going to avail myself of this privilege by discussing three questions that come to my mind. These questions are :—(I) Why was the organization of mathematical research in India so inordinately delayed ? (II) What amount of success has been achieved by the Calcutta Mathematical Society ? (III) To whose efforts and to what circumstances is that success due ?

(I)

Before proceeding to answer the first question, I propose to explain it at some length. In the 17th century and before it, it was not difficult for a great mathematician to be equally strong in every branch of Mathematics. As many of you know, Newton and Leibnitz were not only accomplished mathematicians but had made great contributions to other sciences as well.

But in the 18th century there was a great extension of the bounds of mathematical knowledge, chiefly due to the efforts of Swiss and French mathematicians, and by the middle of the 19th century Mathematics had divided itself into so many important branches that it became almost impossible for one man to explore every one of them. Perhaps the last mathematician to know fully every branch of Mathematics (including Astronomy) was Gauss (1777-1855) who was Professor at the University of Gòttingen for nearly half a century. Even Cauchy (1789-1857), the great French contemporary of Gauss, although a great researcher in Pure as well as Applied Mathematics, did no research work in Astronomy. In the latter half of the 19th century, mathematical knowledge increased with such rapidity that some of the greatest mathematicians remained more or less unacquainted with vitally important branches of Mathematics. I doubt if Henri Poincaré (1854-1912) knew much of the theory of functions of a real variable ; at any rate he did not contribute any research paper to extend the bounds of our knowledge of that theory. Weierstrass's activities were confined chiefly to Pure Mathematics; and his distinguished pupils, Mittag-Leffler, Georg Cantor, Paul du Bois-Reymond and Ulissi Dini confined their research activities entirely to Pure Mathematics.

It will be, therefore, clear to you that the ordinary Societies or Academies of Sciences did not meet the needs of mathematicians. A number of mathematical Societies grew up in Europe and America. Of these the oldest, the Mathematical Society of Hamburg was first founded in 1690 with the name "Kunstrechnungliebende Societät" which was changed in 1790 to "Gesellschaft zur Verbreitung der Mathematischen Wissenschaften" which name was again changed in 1877 to its present name "Mathematische Gesellschaft in Hamburg." The next oldest society is the Mathematisk Forening of Copenhagen. The other societies in order of age are—

(1) London Mathematical Society (1865).

(2) Moscow Mathematical Society (1867).

(3) Societe Mathematique de France (Paris) (1873).

(4) Roland von Eötvös Society for Mathematics and Physics (Budapest) (1878).

(5) Edinburgh Mathematical Society (1883).

(6) Circolo Matematico di Palermo (1884).

(7) American Mathematical Society (first founded as New York Math. Soc.) (1888).

(8) Deutsche Mathematiker-Vereinigung (1890).

(9) Berliner Mathematische Gesellschaft (1901).

(10) Wiener Mathematische Gesellschaft (1904).

To this list may be added the Royal Astronomical Society (1820) although its activities are confined entirely to Astronomy. The Indian Mathematical Society was started as the "Indian Mathematical Club" in 1907 but did not begin publishing its journal before 1909; also, for some years in the beginning, original work did not appear in its journal which contained short notes of an elementary character, problems and their solutions.

My question will now appear to you to be a natural one. Why did not the wave of general mathematical awakening in the period 1865-1890 reach the shores of India? The answer will be found after considering

(*A*) the state of higher mathematical studies and mathematical research in Great Britain;

(*B*) the policy of the Government, and

(*C*) the mentality of the people of India.

(*A*) I quote two great mathematicians to give you an idea of the state of Mathematics in England Professor W. H. Young, a man who is thoroughly familiar with conditions in every country in Europe, says in a book printed in 1918: "In all these (provincial) Universities without exception the standard of Mathematics at Matriculation is extremely low, resembling that in the Little Go at Cambridge, differing only in this that optional papers of somewhat greater difficulty and range are permitted. It is not too much to say that in no part of the civilized world, as far as I am acquainted with, is the standard quite so low as in the Entrance Examinations of the English Universities. The standard at the Little Go is very much the same, on paper at least, as that of the present Matriculation Examination at the University of Calcutta."

"As regards the scope of the Mathematical Tripos itself, it has till quite lately been of such a nature that the most successful candidate is likely to remain in profound ignorance of mathematical discoveries nearly a century old, and I well remember a man who had taken high honours both at Cambridge and at a previous University, devoting the leisure hours of several years subsequent to the time at which he was a Wrangler, to an attempt at obtaining the solution of the general quintic equation by means of radicals. It is possible that he had heard of Galois, but it is certain that he was totally unacquainted both with the theory of Groups, to which Galois' work gave rise, as well as with the eloquently intuitive character of the proof it afforded of solving the general equation of any degree above the fourth by means of radicals." As regards the doctorates, Professor Young says: "At London, and at various provincial Universities, Doctor's degrees are obtainable for Mathematics. But few such degrees are applied for, and the standard required in the original work sent in is by no means always

high. At London a single essay, of moderate merit, has, I am informed, been accepted. Elsewhere much depends on the referees appointed to report."

So much for the state of mathematical studies. Now hear what Prof. G. H. Hardy,* a distinguished English mathematical researcher, says about the state of mathematical research in England: "Mathematics at Cambridge challenges criticism by the highest standards. England is a first-rate country, and there is no particular reason for supposing that the English have less natural talent for Mathematics than any other race; and if there is any first-rate Mathematics in England, it is in Cambridge that it may be expected to be found. We are therefore entitled to judge Cambridge Mathematics by the standard that would be appropriate in Paris or Göttingen or Berlin. If we apply these standards, what are the results? I will state them, not perhaps exactly as they would have occurred to me spontaneously—though the verdict is one which, in its essentials, I find myself unable to dispute—but as they were stated to me by an outspoken foreign friend (a mathematician whose competence nobody could question, and whom nobody could accuse of any prejudice against England, Englishmen or English mathematicians). In the first place, about Newton there is no question; it is granted that he stands with Archimedes or with Gauss. Since Newton, England has produced no mathematician of the very highest rank. There have been English mathematicians, for example, Cayley, who stood well in the front rank of the mathematicians of their time, but their number has been quite extraordinarily small; where France or Germany produces twenty or thirty, England produces two or three. And what have been the peculiar characteristics of such English Mathematics as there has been? Occasional flashes of insight, isolated achievements sufficient to show that the ability is really there, but, for the most part, amateurism, ignorance, incompetence, and triviality."

(*B*) Let us consider what the policy of the Government has been in the matter of the organization of mathematical research in India. Of course, we may ignore the period before, say 1775, during which the Britishers in India were tradesmen or adventurers, out to exploit the resources of the country. It is only when British supremacy was almost unchallenged, that the Britishers took upon themselves the role of administrators. Now, the state of Mathematics in England for 50 years from 1775 to 1825 was deplorable. Even in 1825 it was possible for a man to be Senior Wrangler without knowing a word of Differential or Integral Calculus. Whatever research activities existed in Europe in that period—and those activities were very

* Presidential address at the annual meeting of the Mathematical Association held in March, 1926.

intense—existed on the Continent. Euler, D'Alembert, Lagrange, Laplace, Legendre, Poisson, Poncelet, Fourier, Monge, Gauss, Dirichlet were all of the Continent. If, therefore, there was any hope of the establishment of any organization for mathematical research it could have been realized only with the co-operation of Frenchmen, Swiss or Germans. But naturally such co-operation could not have been permitted by the British Government which even now would not encourage the employment of Europeans of non-British domicile in the Indian Universities

After the year 1825, mathematical studies in England began to improve although even at Cambridge as lat eas 1865, according to Dr. Glaisher, " there was at that time no encouragement to mathematical research. The Sadlerian professorship had, indeed, been founded but though Cayley was always most ready to give assistance to any one who consulted him, he had little concern with actual mathematical instruction in the University, nor was he an inspiring teacher." However, if it had pleased the Government to import British mathematicians of ability it could have done so. There were Indians like Ramchandra of Delhi, who were most anxious to learn Western Mathematics. In fact Ramchandra wrote a book entitled "Maxima and Minima " in 1850, although he had never been at a College. His knowledge of Western Mathematics was self-acquired. The book is professedly free from the notation of the Differential Calculus and was printed first at Calcutta and later in England with an introduction by De Morgan who wanted the book to be circulated in Europe more as a curiosity than as an original contribution intended to extend the bounds of mathematical knowledge.

During the 50 years following the establishment of Universities at Calcutta, Madras and Bombay, the Government imported nearly 50 men of British birth to fill up Professorships of Mathematics at various Government Colleges. Most of these men were raw graduates from British Universities and you can easily conclude from what I quoted from the writings of Professors Young and Hardy what mathematical equipment such a man, as most of them were, could have brought to India. In width and accuracy that equipment was generally inferior to that of a gold medallist of the Calcutta University and very much inferior to that of the juniormost University lecturer in the Department of Pure Mathematics or Applied Mathematics. Yet, among the 50 or so, there were a few men of great ability, who had done good research work at Cambridge and had been Fellows of Colleges there. Why such men accepted employment in India can only be guessed. But it was never their intention to establish schools of research in India or even to encourage real research here. We Indians are wrong in thinking that a Britisher who receives as Professor a salary of Rs. 2,000 a month, *i.e.*, nearly £1,800 a year, a salary nearly double

the salary of Sir Joseph Larmor or David Hilbert, and certainly much more than double the salary which Felix Klein or Henri Poincaré ever received, is guilty of a serious dereliction of duty in not trying to extend the bounds of knowledge. Men like Homersham Cox, G. H. Stuart and T. C. Lewis understood their duties thoroughly and the result was that although they could do good research work here they did nothing of the kind during their stay in India. Of the various Professors appointed by the Government during the period 1857-1907, it can be said that *not more than five or six research papers were published by them taken all together during their stay in India.* Many of them were rewarded by the Government by being raised to the headship of the Education Departments in their provinces. It is therefore clear as daylight that it was not the policy of the Government up to 1907 to encourage the growth of indigenous centres of mathematical research.

(*C*) It may be asked: If the Government did not want the establishment of schools of mathematical research, could not the people of India help themselves unaided by the Government? It will be seen that this is exactly what was done by the mathematicians of Calcutta, headed by the late Sir Asutosh Mookerjee, when in 1908 the Calcutta Mathematical Society was founded. But the mentality of the people of India, even of highly educated people, had become so slavish during the century or more of British rule, that very few men could rise above the prejudices which they had imbibed against Indian graduates, Indian Universities and Indian capacity. It was because of the clear-headedness, the breadth of vision, and dynamic personality of Sir Asutosh that such prejudices could be successfully set aside. Even at the present date, there are highly educated and respected men in Indian Society who are ready to swear by a foreign label, be it the doctor's degree of a British University or the membership of a British Society.

(II)

I take up now the second question: What amount of success has been achieved by the Calcutta Mathematical Society? In answer to this question, let me give you some figures. Up to this date the Society has held over 100 meetings, the total number of papers read exceed 700 and the papers published aggregate to nearly 500. Papers published in the journal of the Society by Indian mathematicians have been quoted as authorities in the Encyklopädie der mathematischen Wissenschaften and in some standard treatises. The number of ordinary members of the Society is about 400 and it exchanges its journal with nearly 90

Societies in India, Europe, North America, Russia, Turkestan, Japan, Australia and South America.

The number of honorary members is 30. The high estimation in which the Society is held all over the world will be clear from the messages which the Secretary will read after my speech and from the fact that thirty eminent mathematicians of Europe, Japan and America have sent in research papers for publication in the journal of the Society.

(III)

The third question: To whose efforts and to what circumstances is this success due ? cannot be completely answered. The success which the Society has achieved is first due to the great interest which Sir Asutosh Mookerjee took in it during the whole of the period of 16 years from 1908-1924. Out of the nearly 75 meetings held in that period, there was only one meeting from which he was absent, possibly due to unexpected illness. The fact that he was not only a mathematician who had dreamt dreams of a career of mathematical research, which because of the attitude of the Government could not materialize, but also the Vice-Chancellor of the Calcutta University for 8 years after the foundation of the Society, enabled the Society to attract sympathy and collaborators which it would otherwise have not succeeded in doing. In the next place, I would attribute our success to the excellent intellectual material which is to be found among Bengal graduates. In my humble opinion, the average Bengali graduate is more idealistic in his outlook than the graduates of other races in India. Coupled with this fact is the existence of a larger number of men who take the Master's degree in Mathematics from year to year at Calcutta. It will be, therefore, clear to you that it is easier to find young men in Bengal who would devote themselves to a career of mathematical study and research. The success of the Post-Graduate Departments of Pure and Applied Mathematics at the Calcutta University is also due to the same cause. In the third place, the success of the Society is due to the devoted labour of love of a succession of treasurers and secretaries. It will be invidious to single out names. But I consider it my duty to state that, but for the devotion of Dr. Bibhutibhushan Datta, who, at the earnest request of Sir Asutosh in 1924, undertook to shoulder the responsibility of the Secretaryship, the Society would have found it extremely difficult to publish so many volumes of its journal as it did in the last 4 years. After Dr. Datta's retirement from the Secretaryship in 1926, the work of that post has been zealously

performed by Dr. N. N. Sen. It should be noted that from the year 1908 up to now the Society has remained quite independent of Governmental help or influence. The only outside body to which we owe gratitude—and I must say that that gratitude is deep indeed—is the Calcutta University. In the last 20 years the Society has been the recipient of generous help from a succession of Vice-Chancellors and Syndicates. In the name of the Calcutta Mathematical Society, I beg to offer my heartfelt thanks to the Calcutta University.

ON CERTAIN PROPERTIES OF NON-ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

By

E. R. Hedrick (*Los Angeles, California*)

(Communicated by Prof. Ganesh Prasad)

[*Read October 8, 1928*]

1. *Introduction.* In a number of recent papers* the theory of functions of a complex variable $z=x+yi$ has been considered in the case in which the function $w=f(z)$ is not analytic in the traditional sense, that is, in which the derivative depends upon the direction in which z approaches zero. We (H. I. W., *loc. cit.*) called such functions *non-analytic*; Kasner (*loc. cit.*) has used that name sometimes, but he has more often used the name *polygenic* to denote such functions.

* Among such papers, I may mention, approximately in the order of their appearance, the following:

Hedrick, Ingold, and Westfall, *Theory of Functions of a Complex Variable.* Journal de Mathématique, ser. 9 (1923), pp. 327-342. I shall refer to this by the abbreviation H. I. W., *loc. cit.*

Hedrick and Ingold, *Analytic Functions in Three Dimensions*, Transactions of the American Mathematical Society, Vol. 27 (1925), pp. 551-555. I shall refer to this as H. I., I, *loc. cit.*

Hedrick and Ingold, *The Beltrami Equations in Three Dimensions*, Transactions of the American Mathematical Society, Vol. 27 (1925), pp. 556-562. I shall refer to this as H.I., II, *loc. cit.*

Edward Kasner, *A New Theory of Polygenic or Non-Monogenic Functions*, Science. Vol. 66 (1927), pp. 581-582. I shall refer to this as K., I, *loc. cit.*

Edward Kasner, *General Theory of Polygenic or Non-Monogenic Functions.—The Derivative Congruence of Circles*, Proceedings of the National Academy of Sciences, Vol. 14 (1928), pp. 75-82. I shall refer to this as K., II, *loc. cit.*

M. Nicolesco, Comptes Rendus, Vol. 185 (1927), p. 442; and Thesis, Paris, 1928.

G. Calugareano, Comptes Rendus, Vol. 186 (1928), p. 186; and Comptes Rendus, Vol. 186 (1928), p. 1406.

Lulu Hoffman and Edward Kasner, Bulletin of the American Mathematical Society, Vol. 34 (1928), pp. 495-503.

Hedrick and Ingold, *Conjugate Functions in Three Dimensions*, Journal de Mathématique, ser. 9, Vol. (1928).

It is my purpose in this paper first to review the grounds for considering such functions, including the question of the proper definition of the word *function*; and then to give several new aspects and new theorems of this theory.

2. *A Discussion of the Definition of the Word Function.*—The theory of functions of a complex variable has been studied chiefly in the case which corresponds to conformal point transformations of the plane, as is shown in all works on the subject. In the theory of functions of real variables, a similar restriction was common some years ago, but it has almost completely disappeared, so that now there is nò such limitation on the word *function* in the case of real variables. The restriction formerly applied in the case of real variables seems different from that still applied to functions of complex variables, but it is seen that the two restrictions are highly analogous if they are stated in the form often used, namely, that *the functions to be treated shall be expansible in Taylor series.* In each case, the limitation arose originally through familiarity with algebraic polynomials, which were then gradually generalized to infinite series of the Taylor form, that is, *power series.* In each case, for many years, many mathematicians refused to recognize functions that could not be thus expressed.

The trend of mathematical thought has been distinctly toward generalization of the original restricted concept. Cauchy discussed this question and gave the classical example $y=e^{-1/x^2}$ of a function whose Taylor series converges but does not represent the function; Fourier dealt with series and with functions not expressible in Taylor form; and the final formulation of a broad and satisfactory definition of the word *function* was given by Dirichlet. During all this period, there was no clean-cut differentiation between functions of real variables and functions of complex variables; but in the recent past the mathematical world seems to have accepted the Dirichlet definition in the case of functions of real variables, but to have clung in the main to the older conceptions in the case of functions of complex variables.

The definition of *function*, essentially as given by Dirichlet, may be stated in modern terms and in a form apparently accepted by the mathematical world, as follows:

A real variable y is said to be a (singled-valued) function of the real variable x, for values of x that belong to a set (E), *if there exists an assignment of values of y to the values of x in* (E), *such that, to every value of x in* (E) *there corresponds a definite value of y.* The function y is then said to be *defined* (and single-valued) *on the range* (E).

Multiply-valued functions, or even infinitely-many-valued functions, may be defined in a similar manner. More generally still, a type of function is recognized in which, corresponding to a definite value (or to a definite

subset) of x in (E), there is a definite set of values of y, which may be any kind of set, even a continuum.

Functions of variables *other than real numbers* have been discussed by many recent writers from a similar standpoint. Volterra and his followers have discussed very extensively *functions of lines*; Frechet and many others have discussed sets of objects in general and functions of them; E. H. Moore and his students have treated functions defined on any range of objects. Even when the dependent variable is a real number, such theories make no attempt to limit the nature of the function, except as hypotheses may be stated in theorems; and the definition of the word *function* that is used is essentially analogous to that of Dirichlet. Still more generally, the dependent variable itself has been taken by several writers to be any variable selected from any domain of objects, so that the idea of *function* has become thoroughly established as the concept of an *assignment* of values of one set of objects to values of another set of objects, with complete freedom, both regarding the sets of objects themselves, and regarding the character of the assignment.

3. *Functions of a Complex Variable: General Formulas.*—We may discuss the theory of functions of a complex variable from a similar standpoint, and such a discussion seems eventually inevitable. A complex variable $z=x+yi$ is determined when the real numbers x and y are given, and conversely. Hence the essential nature of a function of a complex variable is the assignment of values of one such pair (x, y) to values of another such pair (u, v). It has long been recognized, sometimes explicitly,* that the symbol i plays no essential rôle except to distinguish the *first* real number x from the *second* one y. Even the fact that i^2 is taken to be -1 is not in itself significant, but is rather the result of the convenience of the number system that results from the definitions of addition and multiplication.

We shall therefore say that a complex variable $w=u+vi$ (single-valued) is a function $f(z)$ of a complex variable $z=x+yi$, for values of z that belong to a set (E), *if there exists an assignment of values of w to the values of z in* (E), *such that, to every value of z in* (E), *there corresponds a definite value of w.* The function $w=f(z)$ is then said to be *defined* (and single-valued) *on the range* (E).

Generalizations of this definition, as in the case of real variables, are obvious. In accordance with the preceding definition, we may write

$$(1) \quad w=u+vi=f(z)=f(x+yi)=\phi(x, y)+i\psi(x, y),$$

so that

$$(2) \quad u=\phi(x, y), \qquad v=\psi(x, y),$$

where $\phi(x, y)$ and $\psi(x,y)$ are any functions of the real variables x and y. We shall assume in what follows, however, that we are treating only the case in

* See for example, Pierpont, *Theory of Functions of Complex Variables.*

which $\phi(x, y)$ and $\psi(x, y)$, together with their first and second partial derivatives, are continuous. While it is easy to lighten this hypothesis, we shall not take the space below to do so.

The equations (2) define, of course, a point transformation of the plane on to itself. This constitutes no more objection to this definition, however, than does the equally correct statement that the traditional theory is equivalent to the theory of conformal transformations, or than does the statement that the theory of functions of real variables is identically the same as the theory of point transformations of a line on to itself.

The derivative of such a function

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\phi_x + i\psi_x + m\ (\phi_y + i\psi_y)}{1 + im} \tag{3}$$

depends in general on the slope m of the curve on which Δz approaches zero. The result is independent of m, as is shown in all works on the traditional theory, if and only if the Cauchy-Riemann equations

$$\phi_x = \psi_y, \qquad \phi_y = -\psi_x \tag{4}$$

are satisfied, where the subscripts indicate, as usual, partial differentiation.

We (H. I. W., *loc. cit.*) discussed the maximum and minimum values of

$$\rho = \left| \frac{dw}{dz} \right|^2 = \frac{E + 2mF + Gm^2}{1 + m^2} = \frac{d\sigma^2}{ds^2} \tag{5}$$

where ds and $d\sigma$ are the elements of arc in the planes of z and w, respectively, and where

$$E = \phi_x^2 + \phi_y^2, \qquad G = \psi_x^2 + \psi_y^2, \qquad F = \phi_x\phi_y + \psi_x\psi_y. \tag{6}$$

We showed that the values of m which give these maximum and minimum values satisfy the equation

$$F + (G - E)\ m - Fm^2 = 0, \tag{7}$$

and that the maximum and minimum values of ρ itself satisfy the equation

$$\rho^2 - (E + G)\ \rho + J^2 = 0, \tag{8}$$

where

$$(9) \qquad J=\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = \sqrt{EG-F^2}$$

is the jacobian of the transformation (2). We defined the two directions as the *principal directions*, and the curves tangent to them at every point as the *characteristic curves* of the function $f(z)$.

Kasner uses the notation

$$(10) \qquad \gamma=\alpha+i\beta = \frac{dw}{dz}$$

and shows that for a fixed value of z the point (α, β) describes a circle

$$(11) \qquad (\alpha-H)^2 \pm(\beta-K)^2 = h^2+k^2$$

where

$$(12) \qquad 2H=\phi_x+\psi_y, \qquad 2h=\phi_x-\psi_y$$

$$2K=-\phi_y+\psi_x, \qquad 2k=\phi_y+\psi_x$$

Other results of previous papers will be cited in what follows. First of all, however, I shall state certain results which do not involve those ideas.

4. *Implicit Functions*: *Riemann Surfaces*.—It is well-known that such a pair of equations as (2), under the hypotheses already made can be solved for x and y in terms of u and v, near any point $u_0=\phi(x_0, y_0)$, $v_0=\psi(x_0\ y_0)$, if the jacobian (9) does not vanish at (x_0, y_0). On the other hand, if $J(x_0, y_0)=0$, we cannot be sure, without other investigation, that a single-valued solution exists near (u_0, v_0), nor that any solution whatever exists near (u_0, v_0).

If there are several solutions of (2) near (u_0, v_0), they can be represented geometrically by the points of a Riemann surface of several sheets spread over the (u, v) plane, as in the traditional theory of functions. The sheets of the Riemann surface must be connected, if at all, along the curve K in the uv plane which corresponds to the *critical curve* C whose equation is $J(x, y)=0$ in the xy plane. The ordinary *branch points* of the traditional theory are degenerate cases of these lines K of the uv plane, since the jacobian (9) reduces to a sum of squares if the Cauchy-Riemann equations (4) hold,

Any curve

$$(13) \qquad x=p(t), \qquad y=q(t),$$

which passes through a point (x_1, y_1) *at which* $J(x_1, y_1)=0$ *for* $t=t_1$, *corresponds to a curve in the uv plane that is tangent to the curve* K *unless the corresponding point* (u_1, v_1) *is a singular point of* K.

For, the slope of the curve that corresponds to (13) is

$$(14) \qquad \frac{dv}{du}=\frac{\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}}{\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}m}=\frac{\partial\psi}{\partial y}\Big/\frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial x}\Big/\frac{\partial\phi}{\partial x}$$

at (u_1, v_1) since $J(x_1, y_1)=0$, unless all four first partial derivatives ϕ_x, ϕ_y, ψ_x, ψ_y, vanish at (x_1, y_1). And the slope of K is given by finding dv/du from the equations of K :

$$(15) \qquad u=\phi(x, y), \qquad v=\psi(x, y), \qquad J(x, y)=0,$$

which give

$$(16) \qquad \frac{dv}{du}=\frac{\psi_x J_y-\psi_y J_x}{\phi_x J_y-\phi_y J_x}=\psi_y/\phi_y=\psi_x/\phi_x,$$

if $J(x_1, y_1)=0$, unless all four first partial derivatives of ϕ and ψ vanish, or unless both first partial derivatives of J vanish. Since the values given by (14) and (16) agree, the theorem just stated holds, except when C or K has a singular point at (x_1, y_1) or (u_1, v_1).

If the four first partial derivatives of ϕ and ψ do vanish at (x_1, y_1), the slope of the curve corresponding to (13) is

$$(17) \qquad \frac{dv}{du}=\frac{\psi_{xx}\,dx^2+2\psi_{xy}\,dx\,dy+\psi_{yy}\,dy^2}{\phi_{xx}\,dx^2+2\phi_{xy}\,dx\,dy+\phi_{yy}\,dy^2}$$

unless the second derivatives of ϕ and ψ all vanish; and there are, in general, two values of dy/dx that correspond to a single value of dv/du. Thus a single revolution in the xy plane, in this case, corresponds to two revolutions in the uv plane. This is what happens ordinarily at a simple branch point of an analytic function, in the traditional theory. For, if the Cauchy-

Riemann equations (4) hold, the jacobian is a sum of squares, and the curve C consists only of isolated points which are defined by either of the equations

$$(18) \qquad J=\phi_x^2+\phi_y^2=\psi_x^2+\psi_y^2=0, \text{ or } dw/dz=\phi_x+i\psi_x=\psi_y-i\phi_y=0,$$

that is, by

$$(19) \qquad \phi_x=\psi_y=0, \qquad \phi_y=-\psi_x=0$$

so that the branch point is necessarily a singular point of K.

The same fact accounts for the theorem proved independently by Kasner (K., I, *loc. cit.*) that the rate of revolution on the Kasner circle is twice the rate of revolution in the xy plane. This follows from what precedes and from the fact shown in this paper.

Returning to the case in which the point considered is not singular, the critical curve $J=0$ has no singular point at (x_1, y_1). The corresponding curve K of the uv plane then plays a role very analogous to the edge of regression of a developable surface, and I shall call it the *edge of regression* of the function $w=f(z)$. For, we have seen that any curve that meets C corresponds to a curve tangent to K. We may therefore state the theorem :

If any family of curves

$$(20) \qquad x=p\ (t, a), \qquad y=q\ (t, a)$$

cuts the critical curve C for each value of x, the envelop of the corresponding family of curves in the uv plane is precisely this edge of regression K.

This fact results also from a direct calculation of the envelop of the curves corresponding to (20). We shall next show in certain examples that the appearance of the Riemann surface, in many cases, is precisely that of a flattened developable surface, described entirely by the tangents (rectilinear or curvilinear) to this edge of regression K.

5. *Examples of Riemann Surfaces.*—It is easy to construct many examples. One need only write down any point transformation of the type (2) and compute the corresponding jacobian. A particularly important example of this type will be given in § 7.

An example of an elementary nature is given by the equations that define a transformation from rectangular to polar co-ordinates :

$$(21) \qquad u=x\cos y, \quad v=x\sin y,$$

which corresponds to the function

$$(22) \qquad w=f(z)=x(\cos y+i\sin y)=xe^{i\theta},$$

The jacobian is

$$J(x, y)=x.$$

so that the critical curve is the y axis. The edge of regression degenerates, however, into the origin in the uv plane. The arrangement of cuts and sheets is very similar to that for the traditional example

$$w=f_1(z)=e^z=r\ (\cos y+i\sin y).$$

The whole w plane is mapped on the part of the z plane bounded by the lines $x=0$, $y=\pi$, $y=-\pi$, with an obvious division corresponding

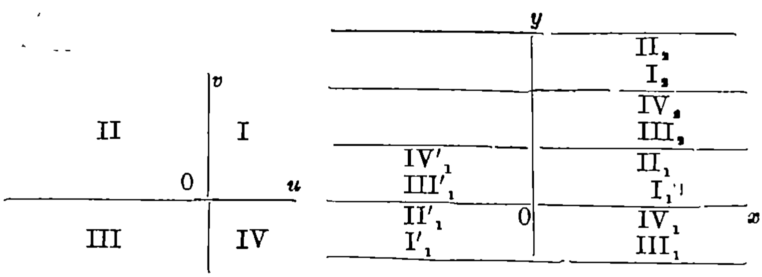


Fig. 1.

to the quadrants of the w plane, as shown in Fig. 1. If z crosses the lines $y=\pm\pi$, there is added a new leaf over the whole w plane, repeating the arrangement, in strips of width 2π, in the z plane. But also, if z crosses the y axis, w will pass across the origin, and will repeat the picture on a new plane that is attached to the previous w plane only at the origin, whereas the arrangement in the z plane is much as before, with the order of quadrants reversed as shown by the marking I'_1, II'_1, III'_1, IV'_1. Thus there are two infinite sets of planes over the w plane, the leaves of each set being connected along the negative u axis in a manner similar to the cut connections for the function $w=e^z$; while the two sets hang together by pairs at the point $w=0$.

In this case, the situation is complicated by the fact that the edge of regression K itself degenerates and at the same time coincides with a point like an ordinary branch point. It is easy, however, to construct examples in which K does not degenerate.

Let us take the curve K arbitrarily in advance, and let us reconstruct an example. Taking for example the parabola

$$(23) \qquad v=u^2$$

to be the edge of regression K, we may draw the straight line tangents to (23). Then, corresponding to any point (u, v) on the parabola or in the region covered by its tangents, we may take x to be the length

$$(24) \qquad x=\int_0^{u_t} u^2\, du=u_t^3/3$$

from the origin to the point (u_t, v_t) of tangency of the tangent through (u, v); and we may take y to be the distance measured in the sense of increasing values of x, along the tangent from (u_t, v_t) to (u, v). The function

$$(25) \quad u=\phi(x,y),\ v=\psi(x,y)\ ; \qquad \text{or} \quad w=f(z)=\phi(x,y)+i\psi(x,y)$$

defined geometrically by what precedes, is obviously single-valued and defined for every point (x, y) of the z plane. It is evident geometrically that the jacobian of this function vanishes on the line $y=0$, which corresponds to the parabola (23). The algebraic calculations are not so easy, but fortunately are not necessary here. It is also evident geometrically that the Riemann surface consists of two leaves over the part of the plane covered by the tangents of the parabola, and that these leaves are connected along the entire length of the parabola.

A similar figure can be constructed beginning with any single curve, using the tangents in a similar manner. Any other family of curves whose envelop is the given curve, may be used in place of the straight-line tangents The connection of such a picture with the solutions of a differential equation which has a singular solution is obvious. The ordinary solutions of the equation may be made to correspond to a set of straight lines parallel to one of the axes in the xy plane, the singular solution is precisely the edge of regression of the function defined, and the distance from some fixed point on it to any other point on it may be selected as the definition of the second co-ordinate in the xy plane.

If the given curve is of class three, that is, if there are three (but no more) tangents from some points to the curve, the resulting Riemann surface

is three-leaved over parts of the plane, the connections between the leaves being along the curve. An interesting example of this sort is given by considering the line co-ordinates * (x, y) of a straight line

$$v+xu+y=0 \tag{26}$$

in the uv plane, corresponding to the cubic equation

$$t^3+\lambda t+\mu=0 \tag{27}$$

where

$$u=t,\ v=t^3\ ;\qquad x=\lambda,\ y=\mu. \tag{28}$$

Then the solutions of the general cubic (27), for given values of x and y, are given by the parameter t which corresponds to the "normal curve" (23), as is shown by Klein. The picture drawn by Klein is thus in reality a three-leaved Riemann surface. It is evident geometrically that two of the leaves are connected along the part of the curve in the first quadrant, and two are connected along the part of the curve in the third quadrant.

6. *The Principal Directions and the Kasner Circle.*—Let us now return to the formulas given in 3. The circle (11) has been called by Kasner *the derivative circle*; I shall call it *the Kasner circle*. Kasner (K., II, p. 80) used the fact that the point $\psi_y+i\psi_x$ lies on this circle, and he gave a beautiful and simple geometric construction for locating the point (α, β) corresponding to any value of m. It is obvious also that the point $\phi_x-i\phi_y$ lies on the circle; and that the two points corresponding to the slopes $m=0$ and $m=\infty$, that is

$$\frac{dw}{dx}=\phi_x+i\psi_x,\ \frac{dw}{dyi}=\psi_y-i\phi_y \tag{29}$$

lie on the circle. The four points just mentioned lie at the corners of a rectangle inscribed in the Kasner circle, as shown in Fig. 2. In that figure, the co-ordinates of C are (H, K), and the rectangle has sides $(2h, 2k)$. Moreover, we have

$$\begin{cases} OA^2=\psi_x{}^2+\psi_y{}^2=G \\ OE^2=\phi_x{}^2+\phi_y{}^2=E \end{cases} \tag{30}$$

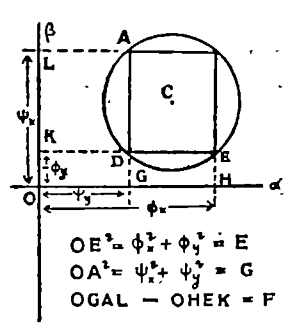


Fig. 2.

* See Klein, Elementarmathematik von höheren Standpunkte aus, Vol. I, p. 97.

If the fundamental quantities E and G are equal, the distances OA and OE are equal, so that AE is perpendicular to OC.

The third fundamental quantity F is given by the formula

$$(31) \qquad F=\phi_x\phi_y+\psi_x\psi_y=\text{Area OGAL}-\text{Area OHEK}.$$

We have therefore in Fig. 2 geometric representations of all of the fundamental quantities E, F, G. If $F=0$, the rectangles OGAL and OHEK are equal in area. If both $E=G$ and $F=0$, however, we have

$$J=\sqrt{EG-F^2}=E=G,$$

and it will appear immediately in what follows that this means that either $OC=0$, or else that $AC=0$, that is, that the Kasner circle either reduces to a point or else has its center at the origin.

I have noted elsewhere * that the maximum and minimum values of $\sqrt{\rho}$ are the maximum and minimum distances (OA and OB in Fig. 3) from the origin to the Kasner circle in the $\alpha\beta$ plane. The principal directions satisfy the equation (7). But, by (3), we have also

$$(32) \qquad m=-\frac{\beta-\psi_x}{\alpha-\psi_y},\ m=\frac{\alpha-\phi_x}{\beta+\phi_y}$$

whence we may show readily that the values of (α,β) corresponding to these principal directions satisfy the equations

$$(33) \quad \begin{cases} -(G-E)(\beta-\psi_x)(\alpha-\psi_y)+F[(\alpha-\psi_y)^2-(\beta-\psi_x)^2]=0, \\ (G-E)(\alpha-\phi_x)(\beta+\phi_y)+F[(\beta+\phi_y)^2-(\alpha-\phi_x)^2]=0. \end{cases}$$

Subtracting, and using (6), we find

$$(34) \quad (G-E)[\alpha(\psi_x+\phi_y)+\beta(\psi_y-\phi_x)]+2F[\alpha(\phi_x-\psi_y)+\beta(\phi_y+\psi_x)]=0,$$

* Hedrick, On derivatives of non-analytic functions, Proceedings of the National Academy of Sciences (1928).

which is a straight line in the $\alpha\beta$ plane through the origin. That the center (H, K) of the Kasner circle lies on this line, appears if we substitute $\alpha=H$, $\beta=K$ in (34). By Fig. 3, we have

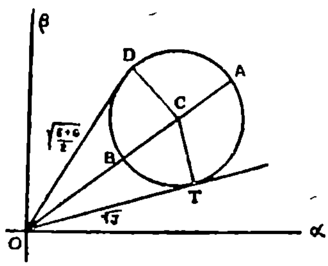


Fig 3

$$\overline{OC}^2=H^2+K^2=\left(\frac{\phi_x+\psi_y}{2}\right)^2+\left(\frac{-\phi_y+\psi_x}{2}\right)^2$$

$$=\frac{\phi_x^2+\phi_y^2+\psi_x^2+\psi_y^2+2[\phi_x\psi_y-\phi_y\psi_x]}{4}$$

$$=\frac{E+G+2J}{4}$$

and similarly

$$\overline{CA}^2=\frac{E+G-2J}{4}$$

whence

$$OA=OC+CA=\tfrac{1}{2}\sqrt{E+G+2J}+\tfrac{1}{2}\sqrt{E+G-2J},$$

$$OB=OC-CA=\tfrac{1}{2}\sqrt{E+G+2J}-\tfrac{1}{2}\sqrt{E+G-2J}.$$

These values do not appear at once to check with the solutions $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$ of (8). Squaring both sides, however, we find

$$\rho_1=\overline{OA}^2=\frac{E+G}{2}+\frac{\sqrt{(E+G)^2-4J^2}}{2},$$

$$\rho_2=\overline{OB}^2=\frac{E+G}{2}-\frac{\sqrt{(E+G)^2-4J^2}}{2},$$

which do check with the solution of (8)

Again referring to Fig. 3, draw the tangent OT to the Kasner circle, and draw CD perpendicular to OC. Then we have

$$(35)\quad \begin{cases} \overline{OD}^2=\overline{OC}^2+\overline{CA}^2=\frac{1}{4}(E+G+2J)+\frac{1}{4}(E+G-2J)=\frac{1}{2}(E+G), \\ \overline{OT}^2=\overline{OC}^2-\overline{CA}^2=\frac{1}{4}(E+G+2J)-\frac{1}{4}(E+G-2J)=J \end{cases}$$

The first equation will be useful in a moment; the last equation is significant, and we may state the following theorem:

The Jacobian $J=\sqrt{EG-F^2}$ is represented geometrically by the square of the tangent from the origin to the Kasner circle; hence also it is equal to the product of the segments of any secant to the Kasner circle from the origin. In particular

$$(36)\qquad \sqrt{\rho_1}\,.\sqrt{\rho_2}=OA.OB=J$$

which agrees with equation (8).

From this theorem it follows immediately geometrically that if $J=0$, the Kasner circle passes through the origin, and conversely. This fact was derived algebraically by Kasner (K, II, *loc. cit.*, p. 79)

It follows also immediately, from the geometry of the figure, that if

$$(37)\qquad E=G,\quad \text{and } F=0,$$

that is,

$$(38)\qquad J=\sqrt{EG-F^2}=E=G=\frac{E+G}{2},$$

the tangent from the origin to the Kasner circle, by (35) must be equal to OD,

$$(39)\qquad OT=OD.$$

But it is evident geometrically that this can happen only either if the Kesner circle reduces to a point, or else if its center is at the origin. These cases correspond to the equations

$$(40)\qquad h^2+k^2=0,\quad \text{or else } H^2+K^2=0,$$

that is, to the equations

$$(41)\qquad \begin{cases}\phi_x=\psi_y,\\ \phi_y=-\psi_x,\end{cases} \text{ or else } \begin{cases}\phi_x=-\psi_y,\\ \phi_y=\psi_x.\end{cases}$$

These are the well-known cases in which the given function $w=f(z)$ is: (*a*) an analytic function of $x+yi$, or (*b*) an analytic function of $x-yi$. This furnishes a geometric proof of the following well-known theorem :

If $E=G$ and $F=0$, that is, if $du^2+dv^2=\lambda^2(dx^2+dy^2)$, the function $w=f(z)$ is either an analytic function of $x+yi$ or else an analytic function of $x-yi$.

Other special cases of interest can be derived geometrically from the values of the distances shown in Figs 2 and 3.

7. *The Jacobian of the Increment-Ratio.*—In another place * I have considered some of the properites of the function

$$(42)\qquad \zeta=\xi+i\eta=\frac{\Delta w}{\Delta z}=\frac{f(z)-f(z_0)}{z-z_0}$$

where $z=z_0$ is the original value of z. This function, which is the increment-ratio for $f(z)$, is itself defined and single-valued for every value of z except for $z=z_0$. I have remarked, but without giving the details of the proof, that the Jacobian of this function necessarily vanishes on the Kasner circle. We shall see that this fact is strongly connected with the facts mentioned in § 4. A direct proof is not difficult; the steps are as follows. There is no essential loss of generality, if we assume that $z_0=0$ and $w_0=f(z_0)=0$, so that the increments of x, y, u, v are simply x, y, u, v. Then

$$(43)\qquad \zeta=\xi+i\eta=\frac{u+vi}{x+yi}=\frac{(ux+vy)+i(vx-uy)}{x^2+y^2},$$

whence

$$(44)\qquad \xi=\frac{ux+vy}{r^2},\qquad \eta=\frac{vx-uy}{r^2},\qquad r^2=x^2+y^2$$

* Bulletin of the American Mathematical Society, Vol. 34 (1928), p. 435,

The Jacobian of ξ and η with respect to x and y is then

$$(45) \qquad J=\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix},$$

where

$$(46) \qquad \xi_x=\frac{(u_x \cdot x+u+v_x y)r^2-(ux+vy)^2 2x_x}{r^4},$$

with similar expressions for ξ_y, η_x, η_y.

Substituting these values, and remembering that

$$ux+vy=\xi r^2, \qquad vx-uy=\eta r^2, \qquad u^2+v^2=(\xi^2+\eta^2)r^2,$$

we find that the equation obtained by setting this Jacobian equal to zero reduces to the form

$$\xi^2+\eta^2-(v_y+u_x)\xi+(v_x-u_y)\eta = -u_x v_y+u_y v_x,$$

which is identical with the equation of the Kasner circle (11). We may therefore state the following theorem :

The Kasner circle is the locus of the equation obtained by equating to zero the Jacobian of the increment-ratio of the function $f(z)$.

As I remarked in § 4, the fact that the Kasner circle is the edge of regression for the increment-ratio function explains why the point (α, β) revolves twice about the circle as the corresponding point in the z plane revolves once about the point in the z plane. For the critical curve in the z plane degenerates to a point, so that we have the case described in § 4. Now the point (α, β) of the Kasner $\alpha\beta$ plane coincides with the point (ξ, η) of the $\xi\eta$ plane along the points of the Kasner circle. Hence the revolution just described corresponds to the fact derived by Kasner for the revolution of the point (α, β). We may combine these statements as follows :

The increment-ratio ζ is a single-valued function of z except for $z=z_0$; its edge of regression on the ζ plane is the Kasner circle, and there are in general two leaves over the ζ plane which are connected along the Kasner circle; as the point (x, y) revolves about the point $z=z_0$, the corresponding point (ξ, η) in general revolves twice about the Kasner circle.

If the function $f(z)$ is analytic at the point $z=z_o$, the expression for ζ becomes

$$\zeta = \frac{f(z)-f(z_o)}{z-z_o} = f'(z_o) + \frac{f''(z_o)}{z}(z-z_o)+\ldots \tag{47}$$

so that

$$\frac{d\zeta}{dz} = \frac{f''(z_o)}{2} + \frac{2f'''(z_o)}{3!}(z-z_o)+\ldots$$

If this derivative vanishes at $z=z_o$, the point to which the Kasner circle degenerates, that is, $\zeta=f'(z_o)$, becomes a branch point, and there are in general two leaves of the Riemann surface for the function which are connected at that point.

THE UNIVERSITY OF CALIFORNIA AT LOS ANGELES,
JULY 4, 1928.

BEMERKUNGEN ZU DEN EXISTENZTHEOREMEN DER KONFORMEN ABBILDUNG

VON

C Carathéodory (*München*)

(*Read, October 8, 1928*)

1. **Einleitung**. Auf den folgenden Seiten sollen zwei wichtige Probleme aus der Theorie der konformen Abbildung mit Hülfe von normalen Familien analytischer Funktionen behandelt werden.

Das zweite dieser Probleme führt zu einem Beweise des Grenzkreistheorems von *Poincaré* und *Koebe*, der mir etwas kürzer und einfacher zu sein scheint als die Beweise dieses Satzes, die mir bekannt sind. Er ist aber nicht grundsätzlich verschieden von diesen.

Über die erste Frage—den Hauptsatz der konformen Abbildung—muss ich einige Worte sagen. Nachdem die Unzulänglichkeit des ursprünglichen Riemannschen Beweises erkannt worden war, bildeten für viele Jahrzehnte die wunderschonen aber sehr umständlichen Beweismethoden, die *H. A. Schwarz* entwickelt hatte, den einzigen Zugang zu diesem Satze. Seit etwa zwanzig Jahren sind dann in schneller Folge eine grosse Reihe von neuen kürzeren und besseren Beweisen vorgeschlagen worden ; [1] es war aber den ungarischen Mathematikern *L. Fejér* und *F. Riesz* vorbehalten auf den Grundgedanken von *Riemann* zurückzukehren und die Losung des Problems der konformen Abbildung wieder mit der Losung eines Variationsproblems zu verbinden. Sie wählten aber nicht ein Variationsproblem, das wie das Dirichletsche Prinzip, ausserordentlich schwer zu behandeln ist, sondern ein solches, von dem die Existenz einer Lösung feststeht. Auf diese Weise entstand ein Beweis, der nur wenige Zeilen lang ist, und der auch sofort in allen neueren Lehrbuchern aufgenommen worden ist.[2] Mein Zweck ist nun zu zeigen, dass man durch eine geringe Modifikation

[1] S. hierzu Z. B. *E. Lindelof* : Sur la representation conforme d'une aire simplement connexe sur l'aire d'un cercle. [4'. Congrés scandinave, Stockholm 1916, p. 59.]

[2] S. *L. Bieberbach* : Lehrbuch der Funktionentheorie, Bd. II, p. 5.

in der Wahl des Variationsproblems den *Fejér-Riesz*schen Beweis noch wesentlich vereinfachen kann.

2. *Zwei Eigenschaften der normalen Familien.* Es sei $\{f(z)\}$ eine Familie von in einem Gebiete G regulären analytischen Funktionen, die in diesem Gebiete *normal* und *kompakt* ist : *d. h.* man soll aus jeder beliebigen Folge $f_1, f_2, \cdots$ von Funktionen der Familie mindestens eine Teilfolge $f_{n1}, f_{n2},$ aussondern können, die in G gegen eine Funktion f_0 "regulär" konvergiert [1] und es soll ausserdem die Grenzfunktion f_0 immer auch zur Familie $\{f\}$ gehören.

Es sei zweitens mit $y\ (f)$ eine Funktionenfunktion bezeichnet, die auf $\{f\}$ definiert und ausserdem endlich und stetig ist. Das heisst zu jeder Funktion $f\ (z)$ aus $\{f\}$ ist eine endliche Zahl $y\ (f)$ zugeordnet, und es ist immer

$$\lim_{n=\infty} y\ (f_n) = y\ (f_0),$$

wenn die Folge f_n regulär gegen f_0 in G konvergiert.

Unter diesen Voraussetzungen hat das Variationsproblem

$$|\ y(f)\ | = \text{Maximum}$$

immer eine Lösung innerhalb der normalen Familie $\{f\}$.

Bezeichnet man in der Tat mit A die ober Grenze aller Zahlen $|\ y(f)\ |$, wenn $f(z)$ die Familie $\{f\}$ durchläuft, so gibt es mindestens eine Folge $f_1, f_2, \ldots$ von Funktionen aus $\{f\}$, für welche

$$\lim_{n=\infty} |\ y(f_n)\ | = \text{A}$$

ist. Nach Voraussetzung kann man aus dieser Folge der f_n eine Teilfolge $f_{n1}, f_{n2}, \ldots$ aussondern, die im Gebiete G regulär gegen eine Funktion $f_0(z)$ konvergiert und, wegen der Stetigkeit der Funktionenfunktion $y\ (f)$, ist

$$|\ y\ (f_0)\ | = \text{A},$$

womit unsere Behauptung bewiesen ist. Gleichzeitig folgt, dass A eine *endliche* Zahl sein muss.

3. Eine seit langem sehr bekannte und sehr leicht zu beweisende Eigenschaft für Folgen von Funktionen $f_n\ (z)$, die in einem Gebiete G meromorph sind und dort gegen eine Funktion $f_0(z)$ regulär konvergieren, ist

[1] *Cf.* eine demnächst in den Mathematischen Annalen erscheinende Arbeit mit dem Titel " Stetige Konvergenz und normale Familien."

folgende: Falls eine ganze Zahl a existiert, derart, dass keine der Gleichungen

$$f_n(z)-a=0 \quad (n=1, 2, \ldots\ldots)$$

mehr als a Nullstellen besitzt, wobei man die mehrfachen Nullstellen vielfach zählt, so ist $f_0(z)$ entweder gleich der Konstanten a oder $f_0(z)$ besitzt auch nicht mehr als a Nullstellen.

Wir nehmen nun an, dass jede der Funktionen $f_n(z)$ das Gebiet G auf ein schlichtes, ganz in Endlichen liegendes Gebiet $\llcorner_n$ abbilden und wollen zeigen, dass $f_0(z)$ dieselbe Eigenschaft besitzt, falls diese Funktion nicht konstant ist.

Wir haben nämlich nur zu beweisen, dass, wenn $f_0(z)$ nicht konstant ist für je zwei verschiedene Stellen z' und z'' in G $f_0(z'')\neq f_0(z')$ ist, oder, was auf dasselbe hinaus kommt, dass der Ausdruck $\{f_0(z)-f_0(z')\}$, als Funktion von z betrachtet, nur die *eine* Nullstelle z' besitzt. Das ist aber nach dem soeben angeführten Satze selbstverständlich, da die Funktionen

$$S_n(z)=f_n(z)-f_n(z')$$

auch nur gerade eine Nullstelle in G besitzen.[1]

4. **Der Hauptsatz der konformen Abbildung.** Es sei G ein *einfach zusammenhängendes* ganz im Endlichen liegendes Gebiet der z-Ebene, das den Punkt $z=0$ in seinem Inneren enthält. Wir wollen zeigen, dass es mindestens eine Funktion

$$w=f_0(z)$$

gibt durch welche G auf den Einheitskreis $|w|<1$ konform abgebildet wird, wobei die Punkte $w=0$ und $z=0$ einander entsprechen sollen.

Wir betrachten die Familie von Funktionen $\{f(z)\}$, die ausser der identisch verschwindenden Funktion $f(z)\equiv 0$ sämtliche Funktionen enthält, die folgenden Bedingungen genügen:

1. Die Funktionen $f(z)$ sind regulär im Gebiete G, und es ist ausserdem $f(o)=0$.
2. In allen Punkten von G ist immer $|f(z)|<1$.
3. Für je zwei verschiedene Punkte z' und z'' von G ist stets

$$f(z')\neq f(z'').$$

Jede Funktion unserer Familie, ausser der einen $f(z)\equiv 0$, liefert also die

[1] Der erste Beweis, den ich von diesem Satze gegeben habe (Math. Ann., Bd. 72, 1912, p. 121) und auch die späteren Beweise, die ich kenne, Z. B. Bieberbach a.a. O., sind wesentlich komplizierter.

konforme Abbildung von G auf ein Gebiet Γ, das im Inneren des Einheits-Kreises $|w|<1$ liegt.

Die Familie $\{f\}$ ist *normal*, weil alle in ihr vorkommenden Funktionen gleichmässig beschränkt sind. Ausserdem ist aber die Familie $\{f\}$ auch *kompakt*, weil jede Grenzfunktion einer innerhalb G regulär konvergierenden Folge von Funktionen aus $\{f\}$ nach dem § 3 entweder konstant ist und daher identisch verschwinden muss, oder allen drei Bedingungen genugt, die die Funktionen der Familie $\{f\}$ charakterisieren.

5. Es sei nun mit z_1 irgend ein fester Punkt innerhalb G bezeichnet der vom Anfangspunkt $z=0$ verschieden ist Wir betrachten die Funktionen-funktion $y(f)$, die durch die Gleichung

$$y(f)=f(z_1)$$

charakterisiert wird. Es ist klar, dass für alle Funktionen aus $\{f\}$ die Funktionenfunktion $y(f)$ beschränkt und im Sinne des § 2 stetig ist. Nach diesem Paragraphen gibt es also eine Funktion $f_0(z)$ unserer Familie, sodass fur jede andere Funktion $f(z)$ aus $\{f\}$ die Relation

$$(5, 1) \qquad |f(z_1)| \leq |f_0(z_1)|$$

bestehen muss.

6. Es ist nun sicher $f_0(z_1)\neq 0$, da fur hinreichend kleine positive Werte der Konstanten a die Funktion

$$f(z) \equiv az$$

allen Bedingungen des § 4 genugt und daher zu $\{f\}$ gehort. Die Funktion

$$(6, 1) \qquad w=f_0(z)$$

liefert also die konforme Abbildung des Gebietes G auf ein Gebiet Γ_0 der w-Ebene, das ganz im Kreise $|w|<1$ liegt und von dem wir beweisen wollen, dass es diesen Kreis ausfullt, womit der behauptete Satz bewiesen sein wird.

Im entgegengesetzten Falle würde mindestens ein Punkt $w=h$ des Randes von Γ_0 in das Innere des Kreises $|w|<1$ fallen.

Es ist ubrigens keine Beschrankung der Allgemeinheit anzunehmen, dass h reell und positiv ist, da mit $f_0(z)$ auch alle Funktionen

$$e^{i\lambda} f_0(z)$$

zur Familie $\{f\}$ gehoren, wenn λ eine reelle Konstante bedeutet : für diese Funktionen ist aber der Wert von $|y(f)|$ derselbe wie fur $f_0(z)$.

Nun betrachten wir die Funktion

$$(6,2)\qquad w=\psi(t)=\frac{h-e^{lh.\frac{1+t}{1-t}}}{1-he^{lh.\frac{1+t}{1-t}}}$$

durch welche die längs des Kreises $|w|=1$ aufgeschnittene und im Punkte $w=h$ logarithmisch verzweigte Riemannsche Fläche auf den Kreis $|t|<1$ derart abgebildet wird, dass die Punkte $w=0$ und $t=0$ einander entsprechen.

Die Umkehrung

$$(6,3)\qquad t=\chi(w)$$

der Funktion (6, 2) ist nun zwar eine unendlich vieldeutige Funktion von w im zweifach zusammenhängenden Gebiete das entsteht, wenn man aus dem Kreise $|w|<1$ den Punkt $w=h$ entfernt. Fur einen Zweig dieser Funktion ist aber $\chi(0)=0$ und wenn man für diesen Zweig die Funktion

$$(6,4)\qquad f(z)=\chi\{f_0(z)\}$$

betrachtet, so ist $f(z)$ regulur im ganzen Gebiete G wenn man diese Funktion längs eines beliebigen Weges innerhalb G fortsetzt, weil ja $|f_0(z)|<1$ und $f_0(z)\neq h$ im diesem Gebiete ist. Ausserdem ist $f(0)=0$, $|f(z)|<1$ und für zwei verschiedene Punkte z' und z'' ist $f(z')\neq f(z'')$. In der Tat ist nach unseren fruheren Resultaten, wenn man $w'=f_0(z')$ und $w''=f_0(z'')$ setzt, $w'\neq w''$ und daher auch im (6, 4)

$$\chi(w')\neq\chi(w''),$$

weil die Umkehrung $\psi(t)$ von $\chi(w)$ eine eindeutige Funktion von t ist. Endlich ist, *weil* G *einfach zusammenhängend ist*, $f(z)$ *eine eindeutige Funktion von z im Gebiete* G. Aus der Gesamtheit dieser Tatsachen folgt, dass $f(z)$ eine Funktion der Familie $\{f\}$ ist.

Setzt man nun $t_1=f(z_1)$ und $w_1=f_0(z_1)$, so hängen diese Zahlen durch die Gleichung

$$(6,5)\qquad w_1=\psi(t)$$

mitenander zusammen. Da nun $\psi(t)$ allen Bedingungen des Schwarzschen Lemmas genugt und da diese Funktion nicht linear ist, folgt aus (6, 5)

$$|w_1|<|t_1|;$$

die letzte Ungleichheit kann auch geschrieben werden

$$|f_0(z_1)|<|f(z_1)|$$

und da sie also der Bedingung (5, 1) widerspricht ist das angekündigte Theorem vollständig bewiesen.

Bemerkung. Im vorhergehenden Beweise hätte man statt mit der Funktion (6, 2) ebensogut mit der Function

$$w=\psi_1(t)=\frac{t(2\sqrt{R}-(1+h)t)}{(1+h)-2\sqrt{R}\,t}$$

arbeiten können. Diese letzte Funktion bildet eine längs des Kreises $|w|=1$ aufgeschnittene und im Punkte $w=h$ verzweigte zweiblättrige Riemannsche Fläche auf den Einheitskreis $|t|<1$ konform ab.

7. **Eigenschaften der konformen Abbildung schlichter Gebiete.** Im Folgenden werden wir zwei Abschätzungen benutzen, die einen Teil des Koebeschen Verzerrungssatzes bilden, aber viel elementarer zu beweisen sind als dieser.

Wir geben diese Sätze, die seit über zwanzig Jahren bekannt sind ohne Beweis an:

Es sei G ein (schlichtes) Gebiet der w-Ebene, das den Punkt $w=0$ aber nicht den Punkt $w=\infty$ in seinem Inneren enthält und bezeichnen mit a die Entfernung des Punktes $w=0$ vom Rande von G. Wir nehmen ferner an, dass durch die Funktion

$$(7,1) \qquad w=\phi(z)$$

das Gebiet G auf den Kreis $|z|<r$ konform abgebildet wird, und dass gleichzeitig die Bedingungen

$$(7,2) \qquad \phi(0)=0 \qquad |\phi'(0)|=1$$

erfüllt sind.

Unter diesen Voraussetzungen gelten erstens die Relationen

$$(7,3) \qquad k.\,z\leq a\leq r,$$

wobei k eine feste Konstante bedeutet, und zweitens ist, falls $0<\delta<1$ und $|z|<\delta r$ genommen wird,

$$(7,4) \qquad |f(z)|<a\mu(\delta),$$

wobei $\mu(\delta)$ eine endliche Funktion von δ bedeutet. Bekanntlich hat *Biebe͏rbach* bewiesen, dass man in (7, 3) die Zahl $k=0{\cdot}25$ setzen kann und dass dies der grösste Wert von k ist, für welchen die Bedingung (7, 3) noch immer gilt.

Man kann aber mit viel einfacheren Mitteln für k den Wert 0, 2 aufstellen und gleichzeitig zeigen, dass man in (7, 4)

$$\mu(\delta)=\delta\, e^{4\frac{1+\delta}{1-\delta}}$$

setzen darf.

8. Aus den Ungleichheiten (7, 3) und (7, 4) folgt nun ohne Weiteres, dass die Familie $\{\phi\}$ der Funktionen $\phi(z)$, die den Kreis $|z|<r$ auf ein schlichtes Gebiet konform abbilden und gleichzeitig den Bedingungen (7, 2) genügen eine normale Familie bilden.

9. **Konvergenzbeweis zum Grenzkreistheorem.** Der Beweis des Grenzkreistheorems von *Poincaré* und *Koebe* kann in mehrere wesentlich verschiedene Teile zerlegt werden. Wenn man nun denjenigen Teil voraussetzt, der mehr zur Topologie als zur Funktionentheorie gehört, so bleibt folgendes Problem zurück:

Wir betrachten eine Folge von unendlich vielen komplexen Ebenen, die wir mit z_1, z_2,...bezeichnen. Auf z_1 liegt ein einfach zusammenhängendes Gebiet $C_1^{(1)}$, das den Punkt $z_1=0$ in seinem Inneren enthält. Auf jeder der übrigen Ebenen z_n ist eine Figur gezeichnet, die aus zwei ineinanderliegenden ebenfalls einfach zusammenhängenden Gebieten $C_n^{(n)}$ und $C_n^{(n-1)}$ besteht: es ist also $C_n^{(n-1)}<C_n^{(n)}$ und wir nehmen ausserdem an, dass der Punkt $z_n=0$ im Inneren von $C_n^{(n-1)}$ liegt. Wir wollen nun zeigen, dass es stets möglich ist auf der komplexen w-Ebene eine unendliche Kette von ineinandergeschachtelten Gebieten $C^{(1)}<C^{(2)}<$...zu konstruieren, derart, dass der Punkt $w=0$ in $C^{(1)}$ enthalten ist, und dass stets bei der konformen Abbildung von $C_n^{(n)}$ auf $C^{(n)}$ bei welcher die Punkte $w=0$ und $z_n=0$ und in diesen parallele Richtungen einander entsprechen, auch das Gebiet $C_n^{(n-1)}$ auf $C^{(n-1)}$ abgebildet wird.

10. Da das Problem nicht modifiziert wird, wenn wir die Figur in der z_n-Ebene durch eine konforme Abbildung irgendwie verändern, bei der die Linien-elemente im Punkte $z_n=0$ fest bleiben, können wir ohne Schaden für die Allgemeinheit voraussetzen, dass alle Gebiete $C_n^{(n)}$ Kreise sind, die durch die Bedingung

$$(10, 1) \qquad |z_n|<r_n$$

festgelegt werden. Ferner können wir die Radien $r_1, r_2, \ldots$ dieser Kreise so wählen, dass eine Funkton

$$(10, 2) \qquad z_n = \phi_n^{(n-1)}(z_{n-1}) \qquad (n=2, 3, \ldots)$$

für welche die Bedingungen

$$(10, 3) \qquad \phi_n^{(n-1)}(0)=0, \; \phi'^{(n-1)}_n(0)=1$$

bestehen, das Gebiet $C_n^{(n-1)}$ auf den Kreis $C_{n-1}^{(n-1)}$ abgebildet wird.

Wegen des Schwarzschen Lemmas gelten dann natürlich die Relationen

$$(10, 4) \qquad r_1 < r_2 < r_3 < \ldots$$

11. Wir führen nun folgende Bezeichnugen ein

$$(11, 1) \quad \begin{cases} \phi_{n+2}^{(n)}(z_n) = \phi_{n+2}^{(n+1)}(\phi_{n+p}^{(n)}(z_n)), \\ \phi_{n+p}^{(n)}(z_n) = \phi_{n+p}^{(n+1)}(\phi_{n+1}^{(n)}(z_n)) \\ (n=1, 2, \ldots, p=3, 4, \ldots). \end{cases}$$

Wir bemerken, dass für alle diese Funktionen die Relationen

$$(11, 2) \qquad \phi_{n+p}^{(n)}(0)=0, \quad \phi_{n+p}'^{(n)}(0)=1$$

von selbst erfüllt sind. Ausserdem sehen wir, wegen der Gruppeneigenschaft der konformen Abbildungen, dass die Funktion $\phi_{n+p}^{(n)}(z_n)$ den Krei $|z_n| < r_n$ auf ein *schlichtes* Gebiet abbildet.

12. Nach dem Resultate des § 8 bilden also die Funktionen $\phi_m^{(n)}(z_n)$ wobei m alle Zahlen, die n übertreffen, durchläuft, für jedes n eine normale Familie. Man kann daher wachsende Folgen $m_1, m_2, \ldots$ von ganzen Zahlen angeben, sodass *gleichzeitig für jeden Wert von* n der Grenzwert

$$(12, 1) \qquad \lim_{j=\infty} \phi_{m_j}^{(n)}(z_n) = f^{(n)}(z_n)$$

existiert und die Konvergenz im Kreise $|z_n| < r_n$ regulär ist.

Da die Funktionen $f^{(n)}(z_n)$ nicht konstant sein können, weil ihre Ableitung im Punkte $z_n=0$ gleich Eins ist, liefern nach dem § 3 die Gleichungen

$$w = f^{(n)}(z_n)$$

eine konforme Abbildung des Kreises $|z_n| < r_n$ auf ein schlichtes Gebiet $C^{(n)}$ der w-Ebene.

13. Wir wollen nun das Bild des Gebietes $C_n^{(n-1)}$ untersuchen, das gleichzeitig mit dem Bilde des Kreises $C_n^{(n)}$ auf die w-Ebene entworfen worden ist. Dieses wird natürlich durch die Funktion

$$\omega = f^{(n)}\left(\phi_n^{(n-1)}\left(z_{n-1}\right)\right)$$

geliefert, wenn z_{n-1} den Kreis $C_{n-1}^{(n-1)}$ durchläuft. Nun bemerke man, dass man für $m_p > n$ nach Definition stets hat

$$\phi_{m_p}^{(n-1)}\left(z_{n-1}\right) = \phi_{m_p}^{(n)}\left(\phi_n^{(n-1)}\left(z_{n-1}\right)\right).$$

Durch Grenzubergang erhält man also

$$^{(n-1)}\left(z_{n-1}\right) = f^{(n)}\left(\phi_n^{(n-1)}\left(z_{n-1}\right)\right),$$

wodurch gezeigt wird, dass das gesuchte Gebiet mit $C^{(n-1)}$ zusammenfallt. Das ım § 9 genannte Problem ist hiermit vollständig gelöst.

14. Es bleibt noch die Gestalt desjenigen Teiles der w-Ebene zu untersuchen, die durch die Kette von Gebieten $C^{(1)}$, $C^{(2)}$, ...ausgefüllt wird. Es sei zunächst

$$\lim_{n=\infty} r_n = \infty;$$

Bezeichnet man mit $a^{(n)}$ den Abstand des Punktes $w=0$ vom Rande des Gebietes $C^{(n)}$ so entnimmt man aus der Relation (7, 3) die Ungleichheit

$$a^{(n)} \geq k\, r_n,$$

aus der folgt, dass $\lim\limits_{n=\infty} a^{(n)} = \infty$ ist, d.h.

dass die Kette der Gebiete $C^{(n)}$ die ganze Ebene überdeckt.

Ist zweitens

$$\lim_{n=\infty} r_n = R,$$

wobei R eine endliche Zahl ist, so folgt, wenn man sich der Bedeutung der

Funktionen $\phi_m^{(n)}(z_n)$ erinnert, dass stets für $|z_n| < r_n$ die Relation

$$|\phi_{m_j}^{(n)}(z_n)| < r_{m_j} < R$$

bestehen muss. Man entnimmt hieraus, dass wenn man zur Grenze übergeht, alle Gebiete $C^{(n)}$ für jeden Wert von n im Inneren des Kreises $|w| < R$ liegen müssen. Die Vereinigung Γ dieser Gebiete ist ein einfach zusammenhängendes Gebiet, von dem wir beweisen wollen, dass es den Kreis $|w| < R$ vollständig ausfüllt.

Im entgegengesetzten Falle würde eine Funktion

$$z=\psi(w) \quad (\psi(0)=0,\ \psi'(0)=1)$$

existiéren, durch welche das Gebiet Γ auf einen Kreis $|r| < R'$ mit $R' < R$ konform abgebildet wird. Wir setzen

$$g^{(n)}(z_n)=\psi(f^{(n)}(z_n))$$

Wegen des Schwarzschen Lemmas ist nun

$$|g'^{(n)}(o)| < \frac{R'}{r_n},$$

und da $g'^{(n)}(0)=1$ ist, folgt hieraus $r_n < R'$ und also im Limes $R \leq R'$, eine Beziehung, die im Widerspruch mit dem früheren steht.

Hieraus folgert man nun leicht, dass die Ränder der Gebiete $C^{(n)}$ gleichmässig den Kreis $|w| = R$ approximieren.

15. Zum Schluss fügen wir eine letzte Bemerkung hinzu: da es keine Funktion $g(z)$ ausser $g^{(z)} \equiv z$ gibt, durch welche mit der Normierung

$$g(o)=0,\ g'(o)=1$$

entweder die ganze Ebene oder der Kreis $|w| \leq R$ auf sich selbst oder auf einen konzentrischen Kreis abgebildet wird, ist die jeweilige Figur, die wir in der w-Ebene erhalten, eindeutig bestimmt.

Man erhält deshalb dieselben Grenzfunktionen $f^{(n)}(z)$, wenn man bei dem Auswahlverfahren des § 12 nicht von den Folgen $\phi_m^{(n)}(z_n)$ selbst, sondern von irgend welchen Teilfolgen dieser Folgen ausgeht. Hieraus folgt insbesondere, dass die Gleichung (12, 1) für jede Folge von Zahlen m_j gelten müssen, also auch z. B. wenn man $m_1=1$, $m_2=2$, u. s. f, setzt.

CERTAIN QUESTIONS IN THE HISTORY OF MATHEMATICS.

By

DAVID EUGENE SMITH (NEW YORK)

(Read July 7, 1929)

There are thousands of questions in the history of every branch of science that remain and must always remain unanswered. The irreparable loss of records, the alteration of manuscripts, the unwarranted claims of writers in periods in which accuracy of statement was subordinated to attractiveness of style, and the natural desire of historians to exaggerate the credit due to their own countrymen are largely responsible for this state of affairs. All this is evident to everyone who has written upon, given courses in, or even read with care the treatises upon the subject. The purpose of this note is not to call attention to any special details, but to mention a few of the larger topics demanding the attention of scholars who are trained in historical research, and in particular in textual criticism.

Chinese writers have in the past, and indeed in recent years, advanced claims which were based upon manuscripts purporting to be copies of various ancient works, these claims tending to show that their countrymen had made noteworthy original contributions to the science of mathematics. This is readily seen in numerous works and articles upon the subject. Two questions are involved : (1) Are the texts as now known reliable ? (2) If so, was the statement (for example, that relating to the early use of $\frac{355}{113}$ for *pi*) original with the writer ? The first shall be capable of answer after a critical examination of the text has been made by highly-trained experts. The second should be studied by equally competent scholars after examining all available sources, particularly Chinese, Indian and Arabian,—the last after

the eighth century. To take a more important case, the unusual interest in the solution of numerical higher equations by Chinese scholars in the 13th century demands particular attention. Are we certain of the reliability of the manuscripts relating to that period,—not because of the opinions of Chinese writers but as a result of the critical study of the texts in accordance with the latest scientific methods? If we find ourselves justified in accepting the texts as they have come down to us, there then arises the question of Arabic influence, and this is a difficult one to answer, but an answer is by no means impossible.

The Japanese problem is not so difficult, for Japan has no such extended history as China; nevertheless there are various important questions awaiting a clear, scientific, unprejudiced reply. For example, we have yet to see a clear translation into any European language of the precise passages from Seki Kōwa that will serve to make clear the validity of the claim that he invented determinants before Leibniz, and that the *yenr's* (circle theory) was a definite step towards a scientific treatment of the infinitesimal calculus. The subject has thus far been too vaguely treated to satisfy historians interested in oriental mathematics.

In recent years the Indian problem has made notable steps towards a solution. On the part of qualified Sanskrit scholars having the necessary mathematical equipment there has developed an interest which is leading to at least a partial answer to the question of the origin of Hindu mathematics. The stimulus of the late Professor Rangacharya's transcription and translation of Mahavira has done much to arouse the recent interest in the subject. Nevertheless there remain the perennial questions of textual criticism and of origin. By way of illustrating what is meant, textual criticism has been able to tell us, with a close degree of approximation to certainty, the epochs in which the various parts of the Hebrew and Christian religious testaments were written. It therefore seems probable that the same degree of care in the study of the Vedas would give us with a fair degree of certainty the dates of the mathematical references which they contain, besides assuring us of the validity of the texts which have come down to us. There is also the important question as to whether or not India borrowed her early mathematics from the Greeks, or whether both Greece and India obtained their early knowledge (as of the approximate value of *pi*) from a common source. The recent excavations in Iraq and the flood of light thereby shed upon the Sumerian civilization give hope that information allowing a fairly definite answer to the question may be forthcoming. This is desirable not merely with reference to the Hindu claims but with respect to those of the Greeks as well

Another field which gives some hope of profitable cultivation is that of the origin of geometric proof. There is a tradition that Thales proved a

few propositions and that his pupil, Pythagoras, demonstrated the theorem which bears his name. The former is somewhat better supported than the latter. Each carries with it the common belief that mathematical demonstration begins with the Greeks, and as yet we have no very tangible historical evidence to the contrary. There are, however, certain conjectures that the Greeks may have had predecessors in the Iraq region—as assuredly they had in astronomy,—or in Egypt or possibly in Crete. There is a tradition that at least one Greek mathematician speaks of the demonstrations of the "harpedonaptae" of Egypt, and a recently-published Moscow manuscript seems to show that the prismoid rule was known to scholars in the Nile valley long before the Dorians were establishing their civilization in Greece. There is therefore a slight reason for the conjecture that some kind of geometric proof may have been known before the Ionian School came into existence. The chance for discovering any historical evidence to sustain this conjecture is small, but there is still hope that one or more papyri may be found that will bear upon the matter.

There are some of the larger problems to be solved, problems that may be called racial or national. Of the more detailed questions it will suffice to mention only two as typical. When Lord Moulton gave his masterly address on Napier, in 1914, he suggested that the latter might have been led to his invention of logarithms by considering some such formula as that of a function of the sum or the difference of two angles. We now know that, before Napier published anything upon the subject, scholars had been doing precisely what Lord Moulton suggested that Napier had in mind; namely, the replacing of multiplication by addition as in these addition formulae. Indeed, a discipline had already been invented, named, and published, some of the particulars of which are set forth in the work on the astrolabe by Clavius. The question then arises as to the origin of this device and as to the steps that led up to its publication. From its nature it might well go back to the Arabic period.

A second illustration may be found in the use of fractional exponents by Oresmus. Had he any predecessors, and did any later writers make any use of his invention? Out of the large number of medieval manuscripts it is possible that one or more may be found that will answer the question.

As to the spread of the Hindu-Arabic numerals in Europe, as to the origin of these numerals, as to the origin of a symbol for zero, as to the early history of the Roman numerals, and as to the place value in notation,—such problems have long attracted the attention of scholars, but the fields have by no means been fully cultivated,—and the same may be said of hundreds of others that should and will be the subject of more critical study than they have as yet received. In particular, an exhaustive treatise upon the development of the calculus awaits the efforts of some able scholar, and so with the questions of

SUR UNE PROPRIÉTÉ DES FONCTIONS À CARRÉ SOMMABLE

PAR

N. Lusin (Moscou)

[*Read December 29, 1928.*]

Dans la Note présente je voudrais indiquer une propriété que chaque fonction $f(\theta)$ à carré sommable possède *presque partout* sans l'avoir, peut-être, *partout*. Cette propriété ne semble pas être triviale puisque il y a une infinité de fonctions *continues* $f(\theta)$ telles que l'ensemble de mesure nulle des points exceptionnels θ se présente effectivement.

I. Considérons une fonction $f(\theta)$ définie dans l'intervalle $(0<\theta<2\pi)$ et à carré sommable, c'est-à-dire telle que l'intégrale

$$\int_0^{2\pi} f^2(\theta)d\theta$$

est finie.

Soit

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\theta+b_n \sin n\theta$$

le développement de la fonction $f(\theta)$ en série de Fourier-Lebesgue. Puisque $f(\theta)$ est une fonction à carré sommable, la série numérique à termes positifs

$$\sum_{n=1}^{\infty} a_n^2+b_n^2$$

est convergente.

Cela posé, désignons par c_n le nombre a_n-ib_n et considérons la série de Taylor

$$F(z)=\frac{c_0}{2}+c_1 z+c_2 z^2+c_3 z^3 \ldots\ldots+c_n z^n+\ldots\ldots,$$

où b_0 est un nombre réel quelconque. Puisqué la série à termes positifs

$$|c_1|^2 + |c_2|^2 + |c_3|^2 + \ldots\ldots + |c_n|^2 + \ldots$$

est convergente, nous avons

$$\lim_{n=\infty} c_n = 0$$

Il en résulte que la fonction $F(z)$ est holomorphe dans l'intérieur du cercle $C(|z|=1)$.

D'ailleurs, on sait que cette fonction $F(z)$ est déterminée *presque partout* sur la périphérie du cercle C. On obtient la détermination $F(\zeta)$ de la fonction $F(z)$ en un point $\zeta = e^{i\theta}$ de la périphérie de C en faisant z tendre vers ζ le long d'un chemin, L intérieur au cercle C et non tangent à sa périphérie : si la limite $F(\zeta)$ de $F(z)$ ne dépend pas de ce chemin L, on prend $F(\zeta)$ pour la détermination de $F(z)$ en ζ.

De plus, on sait que la partie réelle de $F(\zeta)$ coïncide presque partout avec la fonction donnée $f(\theta)$, et la partie imaginaire de $F(\zeta)$ est la fonction à carré sommable $g(\theta)$ conjuguée définie par le développement conjugeé de Fourier—Lebesgue

$$g(\theta) \sim -\frac{b_0}{2} + \sum_{n=1}^{\infty} -b_n \cos n\theta + a_n \sin n\theta.$$

II. Nous allons faire maintenant une remarque relative à le série de Taylor

$$\frac{c_0}{2} + c_1 z + c_2 z^2 + \ldots\ldots + c_n z^n + \ldots$$

Tout d'abord, la fonction

$$W = F(z)$$

de la variable complexe z définie par ce développement ne peut pas être prolongée, dans le cas général, au delà de la circonférence C dans aucun point. Donc, dans le cas général, la périphérie du circle C est la frontiére du domaine naturel d'existence de la fonction $F(z)$.

Si nous faisons varier z dans l'intérieur de C le point W décrit une surface riemannienne S. On sait que l'aire totale de cette surface riemannienne S est donnée par l'intégrale

$$\iint_C |F'(z)|^2 \, dw$$

ètendue sur le cercle C et que la valeur de cette intégrale est

$$|c_1|^2+2|c_2|^2+3|c_3|^2+\cdots+n|b_n|^2+\ldots$$

Donc, si cette série est divergente, l'aire totale de la surface riemannienne S est *infinie* C'est le cas général.

Prenons maintenant une petite partie σ' du cercle C limitée par un petit arc (α, β) de la circonférence C. Si nous faisons varier z dans σ', la variable complexe W décrit une partie correspondante S' de la surface riemannienne S. Comme la circonférence C est, dans le cas général une ligne singuliére essentielle de la fonction $F(z)$, la partie S' de S a une aire *infinie*. Donc, dans le cas général, l'intégrale

$$\iint_{\sigma'} |F'(z)|^2 dw$$

étendue à σ' est *infinie*, quelque soit le petit arc (α, β) de la circonférence C.

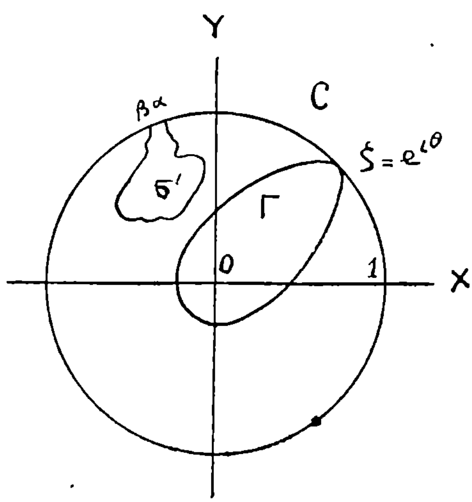


FIG. 1.

Cela posé, *la propriété considérée des fonctions à carré sommalle $f(\theta)$ consiste précisement en ce que presque pour chaque point $\zeta=e^{i\theta}$ de la circonférence C on peut tracer, dans l'intérieur de C, une courbe Γ fermée et tangente en ζ à C telle que l'intégrale*

$$\iint_{\Gamma} | F'(z) |^2 dw$$

étendue à l'intérieur de Γ ait une valeur finie. L'ensemble des points exceptionnels

$\zeta=e^{i\theta}$ de la circonférence C est toujours un ensemble de mesure nulle. D'ailleurs, il y a une infinité de cas dans lesquels l'ensemble des points exceptionnels se présente effectivement, même si la fonction donnée $f(\theta)$ est partout continue sur la circonférence C.

III. Pour démontrer cette proposition nous prenons d'abord la dérivée de la fonction $F(z)$

$$F'(z)=\sum_{n=1}^{+\infty} nc_n z^{n-1},$$

où nous posons

$$z=\rho e^{i\phi}.$$

Nous avons donc, ρ étant inférieur à l'unité

$$F'(z)=\sum_{n=1}^{+\infty} n\rho^{n-1}\left\{\begin{array}{c}[a_n\cos(n-1)\phi+b_n\sin(n-1)\phi]\\ +i\,[a_n\sin(n-1)\phi-b_n\cos(n-1)\phi]\end{array}\right\}$$

et, par suite,

$$|F'(z)|^2=\left(\sum_{n=1}^{+\infty} n\rho^{n-1}[a_n\cos(n-1)\phi+b_n\sin(n-1)\phi]\right)^2$$

$$+\left(\sum_{n=1}^{+\infty} n\rho^{n-1}[a_n\sin(n-1)\phi-b_n\cos(n-1)]\phi\right)^2$$

Puisque ρ est inférieur à l'unité. la formule précédente peut être écrite sous la forme de la somme double

$$|F'(z)|^2=\sum_{m,n}^{1,+\infty} m.n.\rho^{m+n-2}\left\{\begin{array}{c}(a_m a_n+b_m b_n)\cos(m-n)\phi\\ -(a_m b_n-a_n b_m)\sin(m-n)\phi\end{array}\right\}$$

absolument et uniformément convergente.

Soit Γ_θ une courbe fermée, située dans le cercle C et menée par le point $\zeta=e^{i\theta}$ de la circonférence C. Nous ne précisons pas, pour le moment, cette courbe, mais nous la supposons précisée.

Si nous remplaçons, dans la formule précédente, ρ par $t\rho$, t étant un nombre positif inférieur à l'unité et *fixe*, nous pouvons intégrer, terme à terme la somme double précédente dans l'intérieur de la courbe Γ_θ.

En posant

$$dw=\rho d\rho d\phi$$

nous avons donc

$$\iint_{\Gamma_\theta} |F'(z)|^2 dw = \sum_{m,\,n}^{-1,\,+\infty} m.n.t^{m+n-2}$$

$$\times\left\{\begin{array}{l}(a_m a_n + b_m b_n)\,.\displaystyle\iint_{\Gamma_\theta} \rho^{m+n-1}\cos(m-n)\phi d\rho d\phi \\ -(a_m b_n - a_n b_m)\,.\displaystyle\iint_{\Gamma_\theta} \rho^{m-n-1}\sin(m-n)\phi d\rho d\rho\end{array}\right\}$$

Si nous remplaçons, dans les intégrales de la série double, ϕ par $\theta+\phi$, nous pouvons nous borner à la considération d'une courbe fermée unique Γ_0, indépendente de θ, située dans le cercle C et passant par le point $(x=1, y=0)$.

Nous avons de cette maniére

$$\iint_{\Gamma_\theta} |F'(z)|^2 dw = \sum_{m,\,n}^{1,\,+\infty} m.n.t^{m+n-2}$$

$$\times\left\{\begin{array}{l}\left[\begin{array}{l}(a_m a_n + b_m b_n)\cos(m\text{-}n)\theta \\ -(a_m b_n - a_n b_m)\sin(m\text{-}n)\theta\end{array}\right].\displaystyle\iint_{\Gamma_0} \rho^{m+n-1}\cos(m\text{-}n)\phi\, d\rho.\, d\phi \\ \left[\begin{array}{l}-(a_m a_n + b_m b_n)\sin(m\text{-}n)\theta \\ -(a_m b_n - a_n b_m)\cos(m\text{-}n)\theta\end{array}\right].\displaystyle\iint_{\Gamma_0} \rho^{m+n-1}\sin(m\text{-}n)\phi . d\rho d\phi\end{array}\right\}$$

Enfin, n'oublions pas d'observer que la série double est uniformément convergente relativement à θ puisque le nombre positif t est fixe et inférieur à l'unite. Nous pouvons, donc, intégrer cette série double terme à terme par rapport à θ dans l'intervalle total $(0<\theta<2\pi)$.

Nous aurons, donc, finalement

$$(1)\ldots\int_0^{2\pi}\left(\iint_{\Gamma_\theta} |F'(z)|^2 dw\right) d\theta = \sum_{n=1}^{+\infty} n^2 t^{2n-2}(a_n^2 + b_n^2)\iint_{\Gamma_0} \rho^{2n-1} d\rho d\phi.$$

IV. Précisons maintenant le contour Γ_0 : nous le supposons convexe, symétrique par rapport à l'axe réel OX, contenant l'origine des coordonnées à son intérieur et ayant le point ($x=1$, $y=0$) pour point angulaire à tangentes *distinctes*. Soit 2γ l'angle de ces deux tangentes, $0<\gamma<\frac{\pi}{2}$.

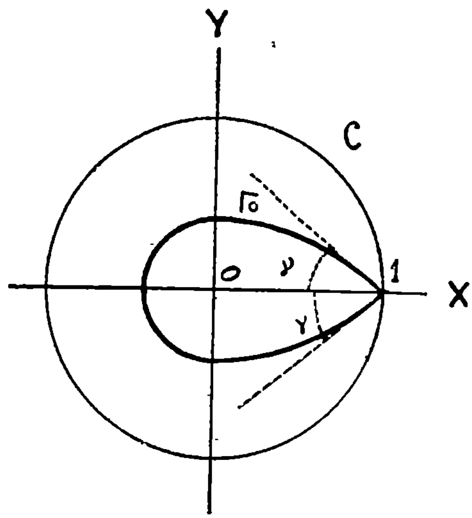


FIG. 2.

Dans ces conditions je dis que *la quantité*

$$n^2\iint\limits_{\Gamma_0} \rho^{2n-1}d\rho d\phi$$

reste bornée quand n croît indéfiniment.

Tout d'abord, en effectuant l'intégration partielle par rapport à ρ, nous avons

$$\frac{n}{2}\int_{-\pi}^{+\pi} \rho^{2n}d\phi.$$

D' autre part, le vecteur $\rho(\phi)$ du contour Γ_0 est manifestement inférieur au vecteur correspondant des deux tangentes au contour Γ_0 issues du point $(x=1, y=0)$ dès que ϕ reste inférieur à un certain nombre positif ϕ_0. Donc, nous avons

$$(2) \quad \ldots \qquad \rho(\phi) < 1 - \frac{\phi}{tg\,\gamma}$$

pour $|\phi| < \phi_0$

Et comme le point $(x=1, y=0)$ est le point *unique* du contour Γ_0 appartenant à la circonférence C, nous pouvons trouver un nombre positif ϕ_1 $\phi_1 < \phi_0$, assez voisin de zéro de manière que nous ayons pour le vecteur ρ du contour Γ_0

$$\rho(\phi) < \rho(\phi_1)$$

quel que soit ϕ restant en déhors de l'intervalle $(-\phi_1, +\phi_1)$

Ceci étant, décomposons l' intégrale précédente en deux parties

$$\frac{n}{2}\left(\int_{-\pi}^{-\phi_1} + \int_{+\phi_1}^{+\pi}\right) \quad \text{et} \quad \frac{n}{2}\int_{-\phi_1}^{+\phi_1} \rho^{2n-1} d\phi.$$

La premiére partie tend nécessairement vers zéro quand n croit indéfiniment puisque sa valeur ne dépasse pas la quantité qu'on obtient en remplaçant $\rho(\phi)$ par $\rho(\phi_1)$, ce qui nous donne

$$\frac{n}{2}\, 2\,(\pi - \phi_1)\,[\,\rho(\phi_1)]^{2n-1} < \pi n\,[\rho\,(\phi_1)]^{2n-1}$$

et puisque, dés que n dépasse un certain entier nous aurons

$$n\,[\,\rho\,(\phi_1)]^{2n-1} < \epsilon$$

car le nombre positif $\rho\,(\phi_1)$ est fixe et inférieur à l' unité.

Il ne reste qu' à évaluer la seconde partie de l'intégrale précédente.

D' aprés la formule (2)

$$(2) \quad \ldots \qquad \rho(\phi) < 1 - \frac{\phi}{tg\gamma}$$

on a

$$\frac{n}{2}\int_{-\phi_1}^{+\phi_1} \rho^{2n-1}\, d\phi < \frac{n}{2}\cdot 2 \cdot \int_0^{+\phi_1} \left(1 - \frac{\phi}{tg\,\gamma}\right)^{2n-1} d\phi$$

$$= \frac{n}{2}\cdot 2 \cdot tg\gamma \cdot \frac{1}{2n}\left[1 - \left(1 - \frac{\phi_1}{tg\gamma}\right)^{2n}\right] < \frac{tg\gamma}{2}.$$

En définitive, la quantité

$$n^2 \iint_{\Gamma_0} \rho^{2n-1} d\phi$$

reste bornée lorsque n croit indéfiniment.

C. Q. F. D.

V. Ces préliminaires terminés, nous allons revenir à l'égalité (1)

$$(1) \qquad \int_0^{2\pi}\left(\iint_{\Gamma_\theta} |F'(z)|^2 dw\right) d\theta = \sum_{n=1}^{+\infty} t^{2n-2}(a_n^2 + b_n^2)\cdot n^2 \iint_{\Gamma_0} \rho^{2n-1} d\rho\, d\phi.$$

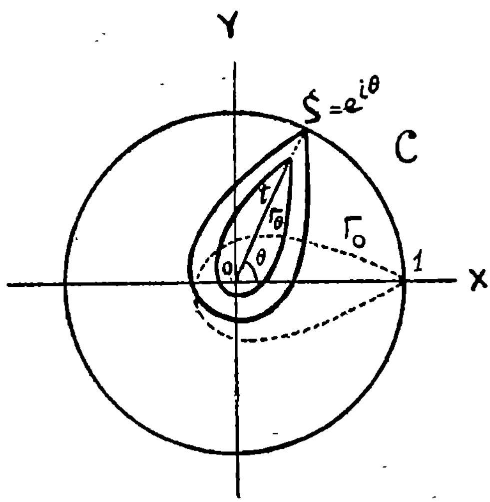


FIG. 3.

Observons d'abord que l'intégrale double

$$\iint_{\theta} |F'(z)|^2 dw$$

est étendue à l'intérieur d'un contour fermée Γ_θ décrit par le point tz, le paramétre positif t étant fixe. Pour obtenir Γ_θ on fait une transformation homothétique de la courbe convexe considérée Γ_0, l'origine des coordonnées O étant pris comme centre et t comme rapport d' homothétie, et l' on tourne ensuite cette transformée de l' angle fixe θ autour de O.

Comme l'origine des coordonnées O est intérieur à Γ_0, il en résulte que, si nous faisons le paramétre t croître et tendre vers l'unitè, l'aire balayée par le contour Γ_θ croît constamment. Donc, *la valeur de l'integral double considérée croît aveo t.*

D'autre part, la série à droite dans la formule (1) ne peut pas croître indéfiniment grand t croît puisque les facteurs.

$$n^2 \iint_{\Gamma_0} \rho^{2n-1} \, d\phi$$

restent bornés et la série $\sum_{n=1}^{\infty} a_n^2 + b_n^2$ converge sûrement.

Donc, la série dans la formule (1) tend vers une limite *finie* lorsque t tend vers l'unité.

On conclut de là immédiatement que l'intégrale simple aux éléments positifs

$$\int_0^{2\pi} \left(\iint_{\Gamma_\theta} | F'(z) |^2 \, dw \right) d\theta$$

est finie pour t=1. De ceci résulte que l'intégrale double

$$\iint_{\Gamma_\theta} | F'(z) |^2 \, dw$$

considérée comme une fonction de la variable θ est *une fonction sommable*, donc *finie presque partout*, lorsquon pose $t=1$.

Nous sommes ainsi amenés à la proposition partielle suivante :

on peut faire correspondre à tout nombre γ positif et inférieur à 2π un ensemble $E\gamma$ de *points de la circonférence C, mes $E\gamma=2\pi$, tel que pour chacun de ses points $\zeta=i\theta$ e l'intégrale double,*

$$\iint_{\Gamma\theta} |F'(z)|^2 dw$$

reste finie, Γ_θ *désignant un triangle isocéle inscrit dans la circonférence C et dont les cotés égaux forment en ζ un angle égal* à γ

Ceci étant établi, faisons tendre l'angle γ vers π par une infinite *dénombrable* de valeurs,

$$\gamma_1 < \gamma_2 < \gamma_3 < \ldots < \gamma_n < \ldots,$$

ou

$$\lim_{n=\infty} \gamma_n = \pi$$

Soit $E\,\gamma_n$ l'ensemble des points situé sur la circonférence C que correspond à l'angle γ_n.

Nous avons toujours

$$\text{mes } E_{\gamma_n} = 2\pi.$$

Comme la partie commune à une infinité dénombrable d' ensembles $E\gamma_n$ dont chacun est de mesure 2π est un ensemble E encore de mesure 2π, nous arrivons au résultat suivant :

on peut définir sur la circonférence C un ensemble de points E de mesure 2π tel que, quelle que soit une courbe $\Gamma\theta$ dans le cercle C et non tangente à sa circonférence C, menée par un point $\zeta=e^{i\theta}$ de E, l'intégrale double

$$\iint_{\Gamma\theta} |F'(z)|^2 dw$$

étendue à l'interieur de Γ_θ ait une valeur finie.

Il est facile maintenant de demontrer la propriété énoncés des fonctions à carré sommable.

En effet, prenons un point $\zeta=e^{i\theta}$ de E. Soit T_n un triangle dans le cercle C dont l'angle au sommet ζ est égal à $\pi-\frac{1}{n}$, et tel que l'intégrale double $\iint_{T_n} |F'(z)|^2 dw$ étendue à l'intérieur de T_n ait ϵ_n pour valeur. Nous supposons la série $\sum_{n=1}^{\infty} \epsilon_n$ convergente.

Désignons par A l'aire couverte par la réunion des triangles T_n, $n=1, 2, 3, \ldots\ldots$. Il est bien évident qu'on peut tracer, dans A, une courbe Γ *tangente* en ζ à la circonférence C.

L'intégrale double.

$$\iint_{\Gamma} |F'(z)|^2 dw.$$

étendue à l'intérieur de Γ a évidemment sa valeur *finie*.

C. Q. F. D.

VI. It serait fort intéressant de pouvoir étendre la propriété obtenue des sèries de Taylor

$$F(z)=\sum_{n=0}^{\infty} c_n z^n$$

ayant la série

$$\sum_{n=0}^{\infty} |c_n|^2$$

convergente.

Pour expliquer aussi clairement que possible en quoi peut consister une telle extension, prenons comme guide une autre propriété de ces series de Taylor que nous avons déjà citée :

On peut déterminer sur la circonférence C un ensemble de points E de mesure 2π tel que, quel que soit un triangle T situé dans le cercle C et ayant un point ζ de E pour sommet, la fonction F (.) tend vers une limite déterminée F (ζ) lors que z tend vers ζ en restant à l'interieur de T.

On sait que, dans, le cas général, la fonction F (ζ) est discontinue en chaque point de la circonférence C, cette discontinuité ne pouvant être écartée par aucun changement de la fonction F (ζ) sur un ensumble de mesure nulle. Et cependant on déduit de la propriété citée de la fonction F(ζ) *l'existence d'une courbe rectifiable fermée* L *située dans le cercle C, ayant avec la circonférence C un ensemble de points communs de mesure aussi voisine de* 2π *que l'on veut, et telle que la fonction* F(z) *soit uniformément continue dans l'aire* L *(contour compris)*.

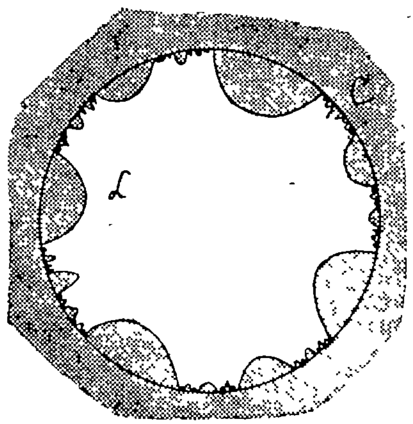


FIG. 4.

La question se pose maintenant de savoir si l'on peut, d'une manière analogue, démontrer *l'existence d'une courbe rectifiable fermée L, situé dans le cercle C et ayant avec la circonférence C un ensemble de points communs de mesure positive, et telle que l'intégrale double*

$$\iint_{L} |\, F'(z) \,|^{2} dw$$

étendue à l'intérieur de L ait une valeur finie ?

C'est une question qui paraît mériter d'attirer l'attention, malgré le doute sur sa réponse positive.

VII. Le problème posé est voisin de l'étude de la convergence des séries de Taylor sur la périphérie du cercle de convergence.

En effet d'après les travaux de M. Féjer, si la série

$$\sum_{n=1}^{\infty} n \mid a_n \mid^2$$

converge, donc si l'intégrale double

$$\iint_C \mid F'(z \mid^2 dw$$

étendue au cercle total C, a une valeur finie, la série de Taylor

$$\sum_1^{\infty} a_n z^n$$

converge presque partout sur la circonférence C.

Or, ainsi que m'a fait remarquer M. Marcel Orbec, il est très probable que cette propriété est *locale* · cela veut dire qui si a_n tend vers zéro avec $1/n$ et si la fonction $F(z)$ reprèsentèe par la sèrie de Taylor

$$\sum_0^{\infty} a_n z^n$$

a l'intégrale double

$$\iint_\Gamma \mid F'(z) \mid^2 dw$$

finie, le domaine Γ étant une partie du cercle C limitée par un arc (α, β) de la circonférence C si petit qu'il soit, la série de Taylor

$$\sum_0^{\infty} a_n z^n$$

converge presque partout sur l'arc (α, β).

C'est la raison pour laquelle il parait très intéressant de savoir s'il y a un lien *direct* ou *indirect* entre l'existence d'une intégrale double finie

$$\iint_\Gamma \mid F'(z) \mid^2 dw$$

étendue à l'intérieur d'une courbe rectifiable fermée ayant avec la circonférence C un ensemble de points communs de mesure *positive* et la convergence presque partout de la série de Taylor dans cet ensemble *

VIII. Parmi les questions de ce genre, it y en beaucoup qui attendent, encore leurs solutions.

Si $F(z)$ est une fonction holomorphe dans le cercle C et uniformément continue dans C (contour compris), l'équation

$$W = F(z)$$

définit, dans le plan de la variable complexe W, une aire A où W varie quand z varie dans le cercle C. Cette aire A est limitée par une courbe jordanienne L, et on obtient tous les points de L en faisant parcourir z un certain ensemble parfait P situé sur la circonférence C. Mais on ne sait pas quelle est la mesure de P.

Dans le cas général où l'on ne suppose que la convergence seule de la série

$$\sum_{0}^{\infty} | c_n |^2,$$

on peut démontrer l'existence de chemins rectifiables L aboutissant à certains points ζ de la circonférence C et tels qu'on obtient encore une courbe rectifiable dans le plan de W en faisant z parcourir L. Mais on ne sait pas quelle est la mesure de l'ensemble E des points de la circonférence C qui correspondent aux points de la surface riemannienne S accessibles par des chemins rectifiables, ou bien rectilignes, ou bien par des angles de grandeur déterminée

IX. Cependant, on peut constater un fait *négatif*: même si la fonction $W = F(z)$ réalise une représentation conforme d'une aire limitée par une courbe jordanienne sur le cercle C ($|z| = 1$), il n'existe pas, en général, dans le cercle C de courbe L fermée et rectifiable ayant sur la circonférence C un ensemble de points de mesure *non nulle* et transformée par l'équation $W = F(z)$ en une courbe encore *rectifiable*.

* Nous n'insistons pas sur ce point, car, sans en méconnaître l'intérêt nous devons constater qu'il est actuellement très probable que la solution complète du problème de la convergence des séries de Fourier viendra de la Théorie des Nombres, ou bien d'un théorème de l'Analyse Mathématique étroitement lié à la Théorie des Nombres.

Pour le voir, prenons sur la circonférence $|W|=1$ un ensemble dénombrable et partout dense de points. Chacun de ces points sera l'origine d'un segment rectiligne dirigé vers le centre du circle, $|W|=1$. Nous appellerons ces segments les *segments du premier rang* et nous choisirons leurs longueurs de telle manière que leur réunion, la circonférence $|W|=1$ ajoutée, forme une courbe jordanienne non rectifiable entre deux points quelconques de la circonférence $|W|=1$.

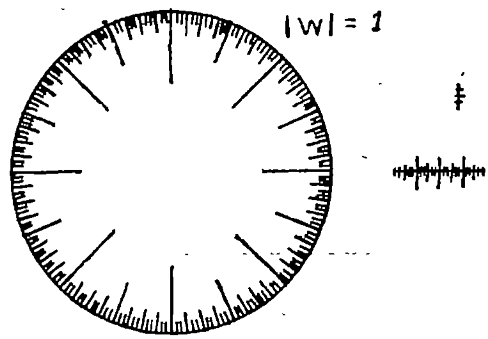


FIG. 5.

Nous prenons ensuite sur chaque segment du premier rang un ensemble dénombrable et partout dense de points et nous élevons, d'une manière symétrique, en ces points des segments perpendiculaires. Ce seront les *segments du second rang*. En choisissant convenablement leurs longueurs nous réussissons à obtenir par la réunion des segments du second rang, du premier rang et de la circonférence $|W|=1$ une courbe jordanienne qui est encore non rectifiable entre deux points quelconques des segments du premier rang, et *ainsi de suite*.

En répétant cette opération, on formera ainsi des segments de tous les rangs finis, les segments de chaque rang étant sans points communs deux à deux.

Un calcul léger montre que si nous ajoutons tous les segments construits à la circonférence $|W|=1$ ainsi que tous les points limites de cette réunion, nous obtiendrons une courbe jordanienne fermée L dans le plan de W telle qu'il n'existe, dans l'intérieur de L (au sens large) aucune courbe rectifiable fermée λ qui ait avec la courbe d'un ensemble de points communs de mesure *non nulle.**)

* C'est M. Weniaminoff qui a établi le premier l'existence des courbes jordaniennes ayant la propriété indiquée (*voir Recueil Mathématique de Moscou*, 1924).

La représentation conforme du domaine limité par la courbe construite L sur le cercle $|z|=1$ nous donne la fonction analytique cherchée $F(z)$ puisque chaque courbe rectifiable z située dans le cercle $|z|=1$ et ayan avec la circonférence $|z|=1$ un ensemble de points communs de mesure non nulle se transforme en une courbe λ non rectifiable.

En effet, supposons la courbe λ rectifiable comme la représentation conforme des domains limités par des courbes rectifiables, l'un sur l'autre, conserve sur leurs frontières les ensembles de *mesure nulle*, on en déduit que l'ensemble des points de z appartenant à la circonférence $|z|=1$ se transforme dans un ensemble de mesure *non nulle*, ce qui est impossible d'après la propriété géométrique indiquée de la courbe construite L.

ON THE FUNCTION θ IN THE MEAN-VALUE THEOREM OF THE DIFFERENTIAL CALCULUS

BY

GANESH PRASAD

[*Read, July 7, 1929*]

The object of the present paper is to investigate, as fully as the present state of Analysis will allow, the nature of the number θ, which occurs in the mean-value theorem

$$f(x+h)=f(x)+h f'(x+\theta h),\ 0<\theta<1, \qquad (M)$$

as a *single-valued* function of h, taking x to be fixed.

Apart from a number of investigations in the last eight years by R. Rothe, T. Hayashi, L. Sokolowski and O. Szász* about the existence of $\theta(+0)$ and $\theta'(+0)$, and the classical result of Dini,† which, strangely enough, finds no mention in those investigations; no attempt seems to have been made by any previous writer at a careful study of the *functional* nature of $\theta(h)$. For this reason, it is hoped, the present paper will advance the knowledge of the properties of $\theta(h)$ with regard to its continuity and differentiability.

For the sake of simplicity, x is taken to be 0 in (M), and the domain of h is taken to be (0, 1). Also, in order to ensure the single-valuedness of θ,

* Rothe : "Zum Mittelwertsatze der Differentialrechnung" (*Math. Zeit.*, Bd. 9, 1921, pp. 300-325) and "Zum Mittelwertsatze und zur Taylorschen Formel" (*Tohoku Math. J.*, Vol. 29, 1928, pp. 145-157).

Hayashi : "On the nature of θ in the Mean Value theorem in Differential Calculus." (*Science Reports of the Tohoku Imperial University*, Ser. I, Vol. XIII, 1925).

Sokolowski : *Tohoku Math. J.*, Vol. 31, 1929, pp. 177-191. Szász : *Math. Zeit.*, Bd, 5, 1926, p. 116.

† "Calcolo Infinitesimale," t, 1, 1907, p. 89 (foot-note),

as will be seen from one of the fundamental theorems, given below, $f'(h)$ must be monotone and continuous. For this reason, throughout the paper

$$f(h)=\int_0^h w(t)\,dt,$$

where w is a monotone, continuous and increasing function. Further, it may be noted that, throughout this paper, by $\psi(t)$ is to be understood such a monotone function in the neighbourhood of $t=0$ that $\psi(t)$ tends to infinity as t tends to 0. Also throughout the paper ξ stands for $h\,\theta(h)$ and $\theta(0)=\theta(+0)$.

In §1 are given a number of fundamental theorems. § 2 contains the condition of Dini and that of R. Rothe for the existence of $\theta(+0)$, and similar conditions for the existence of $\theta'(+0)$. In § 3 and § 4 it is proved by examples that the aforesaid conditions are *not necessary.* § 5 contains sufficient conditions for the existence of $\theta(+0)$ and $\theta'(+0)$ when $f''(0)=0$. In § 6, I give the condition which is *necessary and sufficient* for the existence of $\theta(+0)$, when $w(t)$ is of the form $\int_0^t W(v)\,dv$ and $W(v)$ is equal to a positive function having discontinuities of the second kind only ; for this purpose I take as the type of W

$$W(v)=\chi(v)\left\{2+\sin\psi(v)\right\},$$

where χ is monotone in the neighbourhood of $v=0$ and greater than 0. In §7 the question of the existence of $\theta'(+0)$ is considered and a condition similar to that for $\theta(+0)$ is given. In § 8, I discuss certain types of functions θ each of which is non-differentiable at the points of an everywhere dense set. The paper concludes with § 9 in which, in addition to some remarks on the case of a continuous but *nowhere* differentiable θ, will be found my appreciation and criticism of the investigations of previous writers with reference to the question: What conditions must be satisfied by a prescribed function $\theta(h)$ in order that there should exist a corresponding function $f(h)$?

§ 1

Fundamental Theorems.

1. *Theorem* 1. If θ is a single-valued function of h, then it is necessarily continuous everywhere with the possible exception of $h=0$.

Proof:

Assume that $\bar{h} > 0$ is a point of discontinuity of θ (h). Then, denoting the corresponding values of ξ and θ by $\bar{\xi}$ and $\bar{\theta}$ respectively, we have, by (M),

$$f(\bar{h})=\bar{h}\, f'\ (\bar{\xi}),$$

f (0) being taken to be zero without any loss of generality.

Now, in the modified form of (M), *viz.*,

$$f(h)=hf'(\xi), \qquad (M_1)$$

$f(h)$ is continuous at $\bar{h}$; also $\bar{h}$ is a point of discontinuity of the first kind or a point of discontinuity of the second kind for θ (h) and, consequently, for $\xi(h)$.

(*a*) Let $\bar{h}$ be a point of discontinuity of the first kind. Then, for any sequence $\{h_n\}$ tending to $\bar{h}$ the corresponding sequence $\{\xi_n\}$ does not tend to $\bar{\xi}$ but to $\bar{\xi}'$ *different* from $\bar{\xi}$. But, by (M_1)

$$\frac{f(\bar{h})}{\bar{h}}=f'(\xi),$$

$$\frac{f(\bar{h})}{\bar{h}}=f'(\bar{\xi}').$$

Thus, for the same value $\bar{h}$ of h, there are two different values of ξ, *viz.*, $\bar{\xi}$ and $\bar{\xi}'$, and, therefore, two different values of θ, *viz.*, $\bar{\theta}$ and $\bar{\theta}'$. Hence θ is not single-valued at $h=\bar{h}$, which is against the hypothesis. Therefore it is proved that at any point $h>0$ there cannot be a point of discontinuity of the first kind for $\theta(h)$.

(*b*) Let $\bar{h}$ be a point of discontinuity of the second kind for $\theta(h)$. Then, there must be a sequence $\{h_n\}$ tending to $\bar{h}$, the sequence $\{\xi_n\}$ corresponding to which does not tend to any limit. Therefore, for a neighbourhood of $\bar{h}$ as small as we please, there must be values of ξ, say $\xi_{n_1}, \xi_{n_2}, \ldots \xi_{n_r}, \ldots$ differing from one another by any suitably chosen quantity $\delta>0$. But $f'(\xi_{n_1}), f'(\xi_{n_2}),$ $\ldots f'(\xi_{n_r}) \ldots$ are all different from one another by a quantity as small as we please, because of the continuity of $\dfrac{f(h)}{h}$ at $\bar{h}$; therefore the difference between any two of them must be also as small we please, but this will not

be possible if the difference between any two of the quantities $\xi_{n_1}, \xi_{n_2}, \ldots$ $\xi_{n_r}, \ldots$ is δ. Therefore it is proved that $\bar{h}$ cannot be a point of discontinuity of the second kind for $\theta(h)$.

Combining the conclusions in (*a*) and (*b*) we find that $\theta(h)$ is necessarily continuous for any $h>0$.

Cor. It is an obvious corollary of Theorem 1 that $f'(h)$ must be a continuous function of h in a domain identical with Δ defined below.

2. *Theorem* 2. If $f'(h)$ is monotone and continuous in the domain $(0, 1)$, then there is one-to-one correspondence between h and ξ, h varying in its domain $(0, 1)$ and ξ in its own domain, say Δ.

Proof :

Without loss of generality we may assume that $f'(h)$ is increasing. Thus

$$f(h) = \int_0^h w(t)dt,$$

where w is monotone, continuous and increasing. (*a*) Now, if possible, let there be two values ξ_1 and ξ_2 corresponding to one value h_1 of h. Then, by (M_1),

$$\frac{f(h_1)}{h_1} = f'(\xi_1),$$

$$\frac{f(h_1)}{h_1} = f'(\xi_2).$$

Therefore ξ is not single-valued for the value h_1 of h, which is against our hypothesis. (*b*) Again, let there be two values h_1 and h_2 for one value ξ_1 of ξ. Then, by (M_1),

$$\frac{f(h_1)}{h_1} = f'(\xi_1),$$

$$\frac{f(h_2)}{h_2} = f'(\xi_1).$$

Therefore $F(h) \equiv \frac{f(h)}{h}$ has the same value for two values h_1 and h_2 of h.

But this is impossible as $F(h)$ is a monotone function. For,

$$F'(h) = \frac{hw(h) - f(h)}{h^2}$$

and, because of w being monotone and increasing,

$$f(h) = hw(H),$$

where $H < h$; and hence

$$\frac{hw(h) - f(h)}{h^2} = \frac{w(h) - w(H)}{h} > 0$$

so that $F'(h) > 0$.

Combining the conclusions under (*a*) and (*b*) we have the result that the theorem is proved.

3. *Theorem* 3. If $\theta(h)$ is single-valued and continuous, $\theta'(h)$ need not exist for every value of h.

Proof:

The truth of the above statement is made clear if we take $w(t)$ to be a monotone, continuous and increasing function which has no differential coefficient at the points s of a set S which is everywhere dense in the interval (0, 1). Such a function is, *e.g.*, the function, given by T. Broden in *Crelle's Journal*,* Bd. 118, pp. 27-28, which has a differential coefficient everywhere excepting the points of an everywhere dense set.

Now, by (M_1),

$$f(h) = hf'(\xi) = hw(\xi).$$

As $f(h)$ is differentiable for every value of h, it is also differentiable for a value H_1 corresponding to which $\xi = s_1$, a point of S. Therefore, for $h = H_1$,

$$\frac{d}{dh}\{hw(\xi)\}$$

exists, being $f'(H_1)$. Thus, for $h = H_1$,

$$h\frac{d}{dh}\{w(\xi)\} + w(\xi) = f'(H_1).$$

* See also Hobson's "Theory of functions of a real variable," Vol. 1, 3rd edition, p. 389.

Therefore

$$\frac{d}{dh}\{w(\xi)\} \text{ exists for } h=\mathrm{H}_1.$$

But $\frac{d\xi}{dh}$ cannot exist for $h=\mathrm{H}_1$, for, if it existed, $w'(s_1)$ would exist, being equal to

$$\frac{f(\mathrm{H}_1)-w(s_1)}{\mathrm{H}_1} \div \left(\frac{d\xi}{dh}\right)_{h=\mathrm{H}_1}.$$

Therefore, as $w'(s_1)$ is non-existent, it is proved that ξ, and, consequently, θ are non-differentiable for $h=\mathrm{H}_1$.

4. *Theorem* 4. The function $w(t)$ must not have an infinite number of maxima and minima in the neighbourhood of any point t in the domain Δ.

Proof :

Assume that in the neighbourhood of a point ξ_0 of Δ, there are infinite number of maxima and minima. Then, by a well-known theorem,* there will be infinite number of points $\xi_1, \xi_2, \ldots \xi_r \ldots$ in the neighbourhood of ξ_0 at which $w(\xi)$ has the same value. Now if ξ_1 corresponds to the value h_1 of h in the theorem (M_1), it follows that $\xi_2, \xi_3, \ldots$ are all values which correspond to h_1 as much as ξ_1 ; in other words, $\xi(h)$ and, consequently, $\theta(h)$ have infinite number of values for $h=h_1$. But this is against the hypothesis, that $\theta(h)$ is single-valued. Therefore the assumption, that $w(t)$ has infinite number of maxima and minima in the neighbourhood of any point in Δ, is wrong.

Cor. As a *nowhere* differentiable function has infinite number of maxima and minima in the neighbourhood of every point, it follows from the above theorem that $w(t)$ cannot be a *nowhere* differentiable function.

§ 2

Sufficient conditions for the existence of $\theta(+0)$ or $\theta'(+0)$.

5. *Dini's condition regarding* $\theta(+0)$.—Dini's condition is given by the theorem, that $\theta(+0)$ exists and equals $\frac{1}{2}$, when (*i*), in a finite neighbourhood of $h=0$, $f(h)$ and $f'(h)$ are finite and continuous, and (*ii*) $f''(0)$ exists and is finite and different from 0, without any regard being paid to what is true about $f''(h)$ at the other points of the neighbourhood.

* Hobson, *l.c.*, pp. 876-877.

Proof:

In Cauchy's generalized mean-value theorem,

$$\frac{\phi(x_0+h)-\phi(x_0)}{F(x_0+h)-F(x_0)}=\frac{\phi'(x_0+\theta_1 h)}{F'(x_0+\theta_1 h)},\ 0<\theta_1<1, \qquad (M')$$

put

$$F(x)=(x-x_0),\ \phi(x)=f(x)-f(x_0)-(x-x_0)f'(x_0)-\frac{(x-x_0)^2}{2}f''(x_0).$$

Then (M′) gives

$$f(x_0+h)-f(x_0)-hf'(x_0)-\frac{h^2}{2}f''(x_0)=h\{f'(x_0+\theta_1 h)-f'(x_0)-\theta_1 hf''(x_0)\};$$

$$\begin{aligned} i.e.,\ & f(x_0+h)-f(x_0)-hf'(x_0)-\frac{h^2}{2}f''(x_0)\\ &=h^2\theta_1\left\{\frac{f'(x_0+\theta_1 h)-f'(x_0)}{\theta_1 h}-f''(x_0)\right\} \qquad \dots \qquad (1).\end{aligned}$$

Now $f''(x_0)$ is finite, x_0 being 0; therefore, with h tending to zero, the factor of h^2 in the left side of (1) tends to 0; put it equal to $\frac{\epsilon}{2}$ where ϵ tends to 0 with h.

Thus (1) may be written

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2}\{f''(x_0)+\epsilon\} \qquad \dots \qquad (2).$$

But, by (M),

$$f(x_0+h)=f(x_0)+hf'(x_0+\theta h).$$

Therefore

$$f'(x_0+\theta h)=f'(x_0)+\frac{h}{2}\{f''(x_0)+\epsilon\},$$

$$i.e.,\ \frac{f'(x_0+\theta h)-f'(x_0)}{\theta h}=\frac{1}{2\theta}\{f''(x_0)+\epsilon\}. \qquad \dots \qquad (3)$$

Now, as h tends to 0, the left side of (3) tends to $f''(x_0)$; put it equal to $f''(x_0)+\epsilon_1$ where ϵ_1 tends to 0 with h. Therefore (3) may be written in the form

$$f''(x_0)+\epsilon_1=\frac{1}{2\theta}\{f''(x_0)+\epsilon\} \qquad \dots \qquad (4)$$

As $f''(x_0)\neq 0$ for $x_0=0$, dividing by $f''(x_0)$ both sides of (4) and making h tend to 0, we have $2\theta(+0)=1$, *i.e.*, $\theta(+0)=\frac{1}{2}$.

6. *Rothe's condition regarding* $\theta(+0)$.—Rothe's condition is given by the theorem, that $\theta(+0)$ exists and equals $\frac{1}{2}$, when (*i*), in a finite neighbourhood of $h=0$, $f(h)$, $f'(h)$ and $f''(h)$ are all existent, finite and continuous and $f''(0)$ is different from zero.*

Proof:

By (M),

$$f(h)=f(0)+h\,f'(\xi),\ 0<\xi<h; \quad \dots \quad (1)$$

also, by (M) applied to $f'(\xi)$, we have

$$f'(\xi)=f'(0)+\xi f''(\xi_1),\ \ 0<\xi_1<\xi. \quad \dots \quad (2)$$

Therefore, comparing (1) and (2) we have

$$\xi f''(\xi_1)=\frac{f(h)-f(0)}{h}-f'(0)=\frac{f(h)-f(0)-hf'(0)}{h},$$

i.e., putting θh for ξ,

$$\theta f''(\xi_1)=\frac{f(h)-f(0)-hf'(0)}{h^2} \quad \dots \quad (3)$$

But by (2) of Art. 5, the left side of the above$=\frac{1}{2}\{f''(0)+\epsilon\}$. Also, *as* $f''(h)$ *is continuous and finite* at $h=0$, and ξ_1 tends to 0 with h, $f''(\xi_1)=f''(0)+\epsilon_2$ where ϵ_2 tends to 0 with h.

Thus (3) gives

$$\theta\{f''(0)+\epsilon_2\}=\frac{1}{2}\ \{f''(0)+\epsilon\},$$

from which because of $f''(0)$ being$\neq 0$, we have

$$\theta(+0)=\frac{1}{2}.$$

* See *Math. Zeit.*, Bd. 9, p. 307, or *Tohoku Math. J.*, Vol. 29, pp. 147-148.

7. *Conditions* regarding* $\theta'(+0)$.—If, (*i*) in a finite neighbourhood of $h=0$, $f(h)$, $f'(h)$, $f''(h)$, and $f'''(h)$ are all existent, finite and continuous, and (*ii*) $f''(0)\neq 0$, then

$$\theta'(+0) \text{ exists and equals } \frac{1}{24}\frac{f'''(0)}{f''(0)}.$$

Proof:

$$f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\frac{h^3}{3!}\{f'''(\theta_3 h)\} \quad \dots \quad (1)$$

Also, in (M), putting $x_0=0$,

$$f'(\xi)=f'(0)+\xi f''(0)+\frac{\xi^2}{2}f'''(\theta_4 h). \quad \dots \quad (2)$$

Therefore, comparing (1) and (2), we have from (M),

$$\frac{h^2}{2}f''(0)+\frac{h^3}{6}f'''(\theta_3 h)=h\xi f''(0)+\frac{h\xi^2}{2}f'''(\theta_4 h) \quad \dots \quad (3)$$

Now by Art. 6 or Art. 5, $\xi=h\theta=h\left(\frac{1}{2}+a\right)$, where a tends to 0 with h.

Therefore (4) becomes, on dividing by h^2,

$$\frac{1}{2}f''(0)+\frac{h}{6}f'''(\theta_3 h)=\left(\frac{1}{2}+a\right)f''(0)+\frac{h}{2}\left(\frac{1}{4}+a^2+a\right)f'''(\theta_4 h),$$

$$i.e.,\ af''(0)=\frac{h}{24}f'''(0)+\frac{h}{6}\epsilon_4-\frac{h}{2}\epsilon_5\left(\frac{1}{4}+a^2+a\right)-\frac{h}{2}(a^2+a)f'''(0),$$

putting $f'''(\theta_3 h)=f'''(0)+\epsilon_4$, $f'''(\theta_4 h)=f'''(0)+\epsilon_5$, where ϵ_4, ϵ_5 tend to 0 with h. Now $f''(0)\neq 0$; therefore

$$a=\frac{h}{24}\frac{f'''(0)}{f''(0)}+\mathrm{E},$$

where

$$\mathrm{E}=\frac{h}{f''(0)}\left\{\frac{\epsilon_4}{6}-\tfrac{1}{2}\epsilon_5\left(\tfrac{1}{4}+a^2+a\right)-\tfrac{1}{2}(a^2+a)f'''(0)\right\}.$$

* See Rothe's paper in *Tohoku Math. J.*, Vol. 29, p. 151. A less rigorous proof is given by Hayashi, *l.c.*

Therefore E may be neglected in comparison with

$$\frac{h}{24}\ \frac{f'''(0)}{f''(0)}.$$

as $f'''(0)$ is finite.

Thus it is proved that

$$\operatorname*{Lim}_{h=+0}\frac{\theta-\frac{1}{3}}{h},\ i.e.,\ \theta'(+0)=\frac{1}{24}\ \frac{f'''(0)}{f''(0)}\ .$$

8. If $f'''(0)$ exists and is *finite*, without any regard to what may be the behaviour of $f'''(h)$ at points other than 0, the above result holds. This follows from the result that, with $f^{(n)}\ (x_0)$ existent and finite and the lower differential co-efficients finite and continuous in $(x_0,\ x_0+h)$,

$$f(x_0+h)=f(x_0)\ +\ hf'(x_0)+\frac{h^2}{2!}f''(x_0)$$

$$+\ \dots\ +\frac{h^{n-1}}{(n-1)!}f^{(n-1)}\ \ (x_0)+\frac{h^n}{n!}\left\{f^{(n)}\ (x_0)+\epsilon_n\right\},$$

where ϵ_n tends to zero with h.

Thus the proof given in Art. 7, stands in all essentials.

§ 3

Dini's condition is not necessary for the existence of $\theta(+0)$.

9. That Dini's condition is not necessary, is shown by the following three examples, in the first of which $f''(0)$ is ∞ and still $\theta(+0)$ exists, and in the second and third $\theta(+0)$ exists although $f''(0)$ is ∞ but $f''(h)$ makes infinite number of oscillations when h tends to 0.

***Example* 1** Let $f(h)=h^2\log\dfrac{1}{h}$; then $f''(0)$ exists and, is ∞ and still $\theta(+0)$ exists being $\dfrac{1}{2}$.

Example 2. Let $w(t)=\displaystyle\int_0^t \mathrm{W}\ (v)\ d\,v$ where $\mathrm{W}\ (v)=v^{-\frac{1}{2}}\left\{2+\sin\dfrac{1}{\sqrt{v}}\right\},$

then it is easily seen that

$$w(t)=4t^{\frac{1}{2}}+2t\cos\frac{1}{\sqrt{t}}-2\int_0^t \cos\frac{1}{\sqrt{v}}\,dv$$

$$=4t^{\frac{1}{2}}+2t\cos\frac{1}{\sqrt{t}}+4t^{\frac{3}{2}}\sin\frac{1}{\sqrt{t}}+0\,(t^{2}). \tag{1}$$

Also

$$f(h)\equiv\int_0^h w(t)\,dt=\frac{8}{3}h^{\frac{3}{2}}-4h^{\frac{5}{2}}\sin\frac{1}{\sqrt{h}}+0_1(h^{\frac{5}{2}}). \tag{2}$$

Now, by (M_1), $f(h)=hw(\xi)$. Therefore, from (1) and (2),

$$\frac{8}{3}h^{\frac{3}{2}}-4h^{\frac{5}{2}}\sin\frac{1}{\sqrt{h}}+0_1(h^{\frac{5}{2}})=h\Big\{4\xi^{\frac{1}{2}}+2\xi\cos\frac{1}{\sqrt{\xi}}$$

$$+4\xi^{\frac{3}{2}}\sin\frac{1}{\sqrt{\xi}}+0(\xi^{2})\Big\}$$

$$=h^{\frac{3}{2}}\Big\{4\theta^{\frac{1}{2}}+2h^{\frac{1}{2}}\theta\cos\frac{1}{\sqrt{h\theta}}+4h\theta^{\frac{3}{2}}\sin\frac{1}{\sqrt{h\theta}}+0_2(h)\Big\}.$$

Therefore, dividing both sides by $h^{\frac{3}{2}}$, we have

$$\frac{8}{3}-4h\sin\frac{1}{\sqrt{h}}-4h\theta^{\frac{3}{2}}\sin\frac{1}{\sqrt{h\theta}}+0_3(h)=4\theta^{\frac{1}{2}}+2h^{\frac{1}{2}}\theta\cos\frac{1}{\sqrt{h\theta}}+0_4(h),$$

$$i.e.,\quad \theta^{\frac{1}{2}}=\frac{2}{3}-h\sin\frac{1}{\sqrt{h}}-h.\frac{8}{27}\sin\frac{3}{2\sqrt{h}}-\frac{2}{9}.h^{\frac{1}{2}}\cos\frac{3}{2\sqrt{h}}+0_5(h).$$

Therefore $\theta(+0)$ exists and equals $\frac{4}{9}$, although

$$f''(0)=\operatorname*{Lim}_{h=+0}\frac{w(h)}{h}=\infty.$$

Note that, for $h>0$, $f''(h)=h^{-\frac{1}{2}}\left(2+\sin\frac{1}{\sqrt{h}}\right)$.

Example 3.

$$\text{Let } w(t)=\int_0^t \mathrm{W}(v)dv, \text{ where } \mathrm{W}(v)=v^{-\frac{1}{2}}\left\{2+\sin\frac{1}{\sqrt[3]{v}}\right\}.$$

Then

$$w(t)=4t^{\frac{1}{2}}+3t^{\frac{5}{6}}\cos\frac{1}{\sqrt[3]{t}}+0_6\left(t^{\frac{7}{6}}\right),$$

$$f(h)=\frac{8}{3}h^{\frac{3}{2}}-9h^{\frac{13}{6}}\sin\frac{1}{\sqrt[3]{h}}+0_7\left(h^{\frac{13}{6}}\right).$$

Therefore (M_1) gives

$$\frac{8}{3}h^{\frac{3}{2}}-9h^{\frac{13}{6}}\sin\frac{1}{\sqrt[3]{h}}+0_7\left(h^{\frac{13}{6}}\right)=h\left\{4\xi^{\frac{1}{2}}+3\xi^{\frac{5}{6}}\cos\frac{1}{\sqrt[3]{h}}+0_6\left(\xi^{\frac{7}{6}}\right)\right\}$$

$$i.e.,\quad \frac{2}{3}-\frac{9}{4}h^{\frac{2}{3}}\sin\frac{1}{\sqrt[3]{h}}+0_8\left(h^{\frac{2}{3}}\right)=\theta^{\frac{1}{2}}+\frac{3}{4}h^{\frac{1}{3}}\theta^{\frac{5}{6}}\cos\frac{1}{\sqrt[3]{h\theta}}+0_9\left(h^{\frac{2}{3}}\right).$$

$$\text{Hence}\quad \theta^{\frac{1}{2}}=\frac{2}{3}-\frac{3}{4}\times\left(\frac{2}{3}\right)^{\frac{5}{3}}h^{\frac{1}{3}}\cos\frac{\left(\frac{3}{2}\right)^{\frac{2}{3}}}{\sqrt[3]{h}}+0_{10}\left(h^{\frac{1}{3}}\right).$$

Therefore $\theta(+0)$ exists and equals $\frac{4}{9}$, although

$$f''(0),\ i.e.,\ \operatorname*{Lim}_{h=+0}\frac{w(h)}{h},\ i.e.,\ \operatorname*{Lim}_{h=+0}\left\{4h^{-\frac{1}{2}}+3h^{-\frac{1}{6}}\cos\frac{1}{\sqrt[3]{h}}\right\}\text{ is }\infty.$$

§ 4

Condition of existence and finiteness of $f'''(0)$ not necessary for existence of $\theta'(+0)$.

10. That the condition of the existence and finiteness of $f'''(0)$ is not necessary for the existence of $\theta'(+0)$, is shown by the following three examples, in the first of which $f'''(0)$ is ∞ and still $\theta'(+0)$ exists, and in the second and third $\theta'(+0)$ exists although $f'''(0)$ is ∞ but $f'''(h)$ makes infinite number of oscillations as h tends to 0.

Example 1.

Let $f(h)=h^3 \log\frac{1}{h}$. Then by (M_1)

$$h^3 \log \frac{1}{h}=h \left\{3\xi^2 \log \frac{1}{\xi}-\xi^2\right\}$$

i.e., $\log \frac{1}{h}=3\theta^2 \left(\log \frac{1}{h}+ \log \frac{1}{\theta}\right) -\theta^2.$

Therefore

$$\theta^2=\frac{\log \frac{1}{h}}{3\left(\log \frac{1}{h}+ \log \frac{1}{\theta}\right) -1}=\frac{1}{3} \text{ in the limit when } h \text{ tends to } 0.$$

Now

$$\frac{\theta^2-\frac{1}{3}}{h}=\frac{\frac{1}{3} - \log\frac{1}{\theta}}{h\{3\left(\log \frac{1}{h} + \log \frac{1}{\theta}\right) -1\}}.$$

Therefore

$$\lim_{h=+0} \frac{\theta^2-\frac{1}{3}}{h}, \text{ i.e., } \quad 2\theta(+0)\, \theta'(+0) =0.$$

Thus $\theta'(+0)$ exists although $f'''(0)=\infty$.

Example 2.

$$\text{Let } w(t)=\int_0^t \mathrm{W}(v)dv,\ \mathrm{W}(v)=2+\int_0^v \mathrm{Y}(u)du$$

and

$$\mathrm{Y}(u)=u^{-\frac{1}{2}} \left\{2+ \sin\frac{1}{\sqrt{u}} \right\}.$$

Then it can be easily seen that

$$\mathrm{W}(v)=2+4v^{\frac{1}{2}}+2v\cos\frac{1}{\sqrt{v}}+4v^{\frac{3}{2}}\sin\frac{1}{\sqrt{v}}+0\left(v^{\frac{3}{2}}\right),$$

$$w(t)=2t+\frac{8}{3}t^{\frac{3}{2}}-4t^{\frac{5}{2}}\sin\frac{1}{\sqrt{t}}+0_1\left(t^{\frac{5}{2}}\right),$$

$$f(h)\equiv\int_0^h w(t)dt=h^2+\frac{16}{15}h^{\frac{5}{2}}-8h^4\cos\frac{1}{\sqrt{h}}+0_{10}\left(h^{\frac{7}{2}}\right).$$

Therefore, by (M_1),

$$h^2+\frac{16}{15}h^{\frac{5}{2}}-8h^4\cos\frac{1}{\sqrt{h}}+0_{10}\left(h^{\frac{7}{2}}\right)=h.\left\{2\xi+\frac{8}{3}\xi^{\frac{3}{2}}\right.$$

$$\left.-4\xi^{\frac{5}{2}}\sin\frac{1}{\sqrt{\xi}}+0_1\left(\xi^{\frac{5}{2}}\right)\right\};$$

i.e.,

$$1+\frac{16}{15}h^{\frac{1}{2}}-8h^2\cos\frac{1}{\sqrt{h}}+0_{11}\left(h^{\frac{3}{2}}\right)=2\theta+\frac{8}{3}h^{\frac{1}{2}}\theta^{\frac{3}{2}}$$

$$-4h^{\frac{3}{2}}\theta^{\frac{5}{2}}\sin\frac{1}{\sqrt{h\theta}}+0_{12}\left(h^{\frac{3}{2}}\right).$$

Hence $\theta\ (+0)=\frac{1}{2}$, and

$$2\theta=1+h^{\frac{1}{2}}\left\{\frac{16}{15}-\frac{4}{3\sqrt{2}}\right\}+\frac{1}{\sqrt{2}}h^{\frac{3}{2}}\sin\frac{\sqrt{2}}{\sqrt{h}}+0_{13}\left(h^{\frac{3}{2}}\right).$$

Therefore

$$\operatorname*{Lim}_{h=+0}\frac{\theta-\frac{1}{2}}{h}=\left\{\operatorname*{Lim}_{h=+0}\frac{h^{\frac{1}{2}}}{h}\right\}\times\text{constant}=\infty.$$

Thus $\theta'(0)$ exists and is ∞ ; also

$$f'''(0)=\operatorname*{Lim}_{h=+0}\frac{\mathrm{W}(h)-2}{h}=\infty.$$

Note that for $h>0$, $f'''(h)=\Upsilon(h)=h^{-\frac{1}{2}}\left\{2+\sin\frac{1}{\sqrt{h}}\right\}$.

Example 3. Let $w(t)=\int_0^t W(v)dv$, $W(v)=2+\int_0^u Y(u)du$ and

$$Y(u)=u^{-\frac{1}{2}}\left\{2+\sin\frac{1}{\sqrt[3]{u}}\right\}.$$

Then it can be easily seen that

$$W(v)=2+4v^{\frac{1}{2}}+3v^{\frac{5}{6}}\times\cos\frac{1}{\sqrt[3]{v}}+0_6\left(v^{\frac{5}{6}}\right),$$

$$w(t)=2t+\frac{8}{3}t^{\frac{3}{2}}-9t^{\frac{11}{6}}\sin\frac{1}{\sqrt[3]{t}}+0_7\left(t^{\frac{13}{6}}\right),$$

$$f(h)=h^2+\frac{16}{15}h^{\frac{5}{2}}-27h^{\frac{19}{6}}\cos\frac{1}{\sqrt[3]{h}}+0_{14}(h^{\frac{19}{6}}).$$

Therefore, by (M_1), the above $=h\{2\xi+\frac{8}{3}\xi^{\frac{3}{2}}-9\xi^{\frac{13}{6}}\sin\frac{1}{\sqrt[3]{\xi}}+0_7(\xi^{\frac{13}{6}})\}$

i.e., $1+\frac{16}{15}h^{\frac{1}{2}}-27h^{\frac{7}{6}}\cos\frac{1}{\sqrt[3]{h}}+0_{15}(h^{\frac{7}{6}})=2\theta+\frac{8}{3}h^{\frac{1}{2}}\theta^{\frac{3}{2}}$

$$-9h^{\frac{7}{6}}\sin\frac{1}{\sqrt[3]{h\theta}}.$$

Hence

$$2\theta=1+\left(\frac{16}{15}-\frac{4}{3\sqrt{2}}\right)h^{\frac{1}{2}}+0_{16}(h),\text{ and, consequently, }\theta'(0)=\infty.$$

Also $f'''(0)=\operatorname*{Lim}_{h=+0}\frac{W(h)-2}{h}=\infty.$

Note that, for $h>0$, $f'''(h)=Y(h)=h^{-\frac{1}{2}}\left\{2+\sin\frac{1}{\sqrt[3]{h}}\right\}$;

§5

Sufficient conditions for the existence of $\theta(+0)$ and $\theta'(+0)$ when $f''(0)$ is zero.

11. *Condition similar to Dini's for $\theta(+0)$.*—Even if $f''(0)$ is zero, $\theta(+0)$ exists and equals $\frac{1}{\sqrt{3}}$ when (*i*), in a finite neighbourhood of $h=0$, $f(h)$, $f'(h)$ and $f''(h)$ are finite and continuous and (ii) $f'''(0)$ exists, is finite and different from zero.

Proof :

From the theorem mentioned in Art. 8, it follows that

$$f(h)=f(0)+h\,f'(0)+\frac{h^3}{3!}\,\{f'''(0)+\epsilon_6\}, \qquad \dots \quad (1)$$

where ϵ_6 tend to 0 with h. Thus, by (M_1),

$$h\,f'(\xi)=h\,f'(0)+\frac{h^3}{3!}\,\{f'''(0)+\epsilon_6\} \qquad \dots \quad (2)$$

But, applying to $w(\xi)\equiv f'(\xi)$ the theorem mentioned in Art. 8, we have

$$w(\xi)=w(0)+\xi.\,w'(0)+\frac{\xi^2}{2!}\,\{w''(0)+\epsilon_7\},$$

$$i.e.,\ f'(\xi)=f'(0)+\frac{\xi^2}{2!}\,\{f'''(0)+\epsilon_7\}. \qquad \dots \quad (3)$$

From (2) and (3), we have

$$\frac{h^3}{3!}\{f'''(0)+\epsilon_6\}=\frac{h\,\xi^2}{2!}\,\{f'''(0)+\epsilon_7\},$$

$$i.e.\ \frac{h^3}{3}\,\{f'''(0)+\epsilon_6\}=h^3\theta^2\{f'''(0)+\epsilon_7\}.$$

Hence, as $f'''(0)\neq 0$, θ^2 tends to $\frac{1}{3}$ as h tends to 0. Thus it is proved that $\theta(+0)$ exists and equals $\frac{1}{\sqrt{3}}$.

12. *Condition for* $\theta'(+0)$.—Even if $f''(0)=0$, $\theta'(0)$ exists and equals $\left(\frac{1}{8\sqrt{3}}-\frac{1}{18}\right)\frac{f^{\text{iv}}(0)}{f'''(0)}$, when (*i*), in a finite neighbourhood of $h=0$, $f(h)$, $f'(h)$, $f''(h)$ and $f'''(h)$ are finite and continuous, and (*ii*) $f^{\text{iv}}(0)$ exists, is finite and different from 0.

Proof :

By using the theorem, mentioned in Art. 8, to expand $f(h)$ and $w(\xi)$, we have

$$f(h)=\frac{h^3}{3!}f'''(0)+\frac{h^4}{4!}\{f^{\text{iv}}(0)+\epsilon_8\},$$

$$w(\xi)=\frac{\xi^2}{2!}w''(0)+\frac{\xi^3}{3!}\{w'''(0)+\epsilon_9\},$$

where ϵ_8 and ϵ_9 tend to 0 with h.

Thus, using (M_1), and remembering that $w''(0)=f'''(0)$, $w'''(0)=f^{\text{iv}}(0)$, we have

$$\frac{f'''(0)}{3!}+\frac{h}{4!}f^{\text{iv}}(0)=\frac{\theta^2}{2!}f'''(0)+\frac{\theta^3 h}{3!}f^{\text{iv}}(0)+0_{17}(h).$$

Therefore

$$\theta^2=\frac{1}{3}+\frac{hf^{\text{iv}}(0)}{f'''(0)}\left\{\frac{1}{12}-\frac{1}{9\sqrt{3}}\right\}+0_{18}(h).$$

Therefore

$$\lim_{h\to+0}\frac{\theta^2-\frac{1}{3}}{h}=\left(\frac{1}{12}-\frac{1}{9\sqrt{3}}\right)\frac{f^{\text{iv}}(0)}{f'''(0)},$$

$$i.e.,\ 2\theta(+0)\theta'(+0)=\left(\frac{1}{12}-\frac{1}{9\sqrt{3}}\right)\frac{f^{\text{iv}}(0)}{f'''(0)};$$

$$\text{so}\ \ \theta'(+0)=\frac{\sqrt{3}}{2}\left(\frac{1}{12}-\frac{1}{9\sqrt{3}}\right)\frac{f^{\text{iv}}(0)}{f'''(0)}=\left(\frac{1}{8\sqrt{3}}-\frac{1}{18}\right)\frac{f^{\text{iv}}(0)}{f'''(0)}.$$

§ 6

Certain general types of $f(h)$ with the necessary and sufficient conditions for the existence of $\theta(+0)$ for each type.

13. *Type I* : $w(t)=\int_0^t \{2+\sin\psi(v)\}dv.$

$\theta(+0)$ exists or does not exist according as

$$\psi \succ \log\frac{1}{v}$$

or

$$\preceq \log\frac{1}{v}\,.$$

Proof:

(*a*) Let $\psi \succ \log\dfrac{1}{v}$. Then

$$w(t)=2t+\int_0^t \sin\psi(v)dv$$

$$=2t-\frac{\cos\psi(t)}{\psi'(t)}+0_{18}(t).$$

Also

$$f(h)=\int_0^h w(t)dt=h^2-\int_0^h \frac{\cos\psi(t)dt}{\psi'(t)}+0_{19}(h^2)$$

$$=h^2-\frac{\sin\psi(h)}{\{\psi'(h)\}^2}+0_{20}(h^2)$$

$$=hw(\xi)$$

[by (M_1)]

$$=h\left\{2\xi-\frac{\cos\psi(\xi)}{\psi'(\xi)}\right\}+h\,0_{18}(\xi).$$

Dividing both the sides of the equation,

$$h^2 - \frac{\sin\psi(h)}{\{\psi'(h)\}^2} + 0_{20}(h^2) = h^2 \left\{2\theta - \frac{\cos\psi(h\theta)}{h\psi'(h\theta)}\right\} + h\, 0_{18}(h\theta)$$

by h^2 and taking the limit when h tends to 0, we have

$$2\,\theta(+0)=1, \text{ i.e., } \theta(+0)=\frac{1}{2}.$$

(*b*) Let $\psi \asymp \log \frac{1}{v}$. Then it can be proved without difficulty * that

$$w(t)=2t+\mathrm{A}t\cos\{\psi(t)+\mathrm{B}\}+0(1),$$

where A is a constant different from zero and B is another constant.

Also $f(h)=h^2+\mathrm{A}_1 h^2 \cos\{\psi(h)+\mathrm{B}_1\}+0(h)$, A_1 being a constant$\neq 0$ and B_1 another constant.

Therefore, by (M_1), $\theta(+0)$ is non-existent as $\lim\limits_{h=0} \dfrac{1+\mathrm{A}_1\cos\{\psi(h)+\mathrm{B}_1\}}{2+\mathrm{A}\cos\{\psi(h\theta)+\mathrm{B}\}}$ is non-existent, because if the limit $\theta(+0)$ existed $\lim\limits_{h=0} \dfrac{1+\mathrm{A}_1\cos\{\psi(h)+\mathrm{B}_1\}}{2+\mathrm{A}\cos[\psi\{h\theta(+0)\}+\mathrm{B}}$ would exist which is not possible as $\theta(+0)$ cannot be 1.

14. *Type II*: $w(t)=\int_0^t \chi(v)\{2+\sin\psi(v)\}\,dv$, $\chi \succ 1$.

$\theta(+0)$ exists or does not exist according as

$$\frac{\chi}{\psi'} \prec v$$

or

$$\frac{\chi}{\psi'} \succeq v.$$

* See my paper, "On the differentiability of the integral function" (*Crelle's Journal*, Bd. 160, pp. 103 and 104).

Proof :

(a) Let $\frac{\chi}{\psi'} \prec v$. Then, denoting $\int_0^t \chi(v)\,dv$ by $X_1(t)$,

$$w(t)=2X_1'(t)-\frac{\chi(t)}{\psi'(t)}\cos\psi(t)+0_{21}(t),$$

$$f(h)=2X_2(h)-\frac{\chi(h)}{\{\psi'(h)\}^2}\sin\psi(h)+0_{22}(h^2),$$

where $X_2(h)$ denotes $\int_0^h X_1(t)dt$.

But $f(h)=h\,w(\xi)$ by (M_1).

Therefore, dividing both the sides of the above equation by $2X_2(h)$, we have

$$1-\frac{\chi(h)}{2X_2(h)\{\psi'(h)\}^2}\sin\psi(h)+\frac{0_{22}(h^2)}{2X_2(h)}=\frac{h\,X_1(h\theta)}{X_2(h)}-\frac{h\,\chi(h\theta)}{2X_2\psi'(h\theta)}\cos\psi(h\theta)$$

$$+\frac{h\,0_{21}(h}{2X_2(h)}.$$

Now let h tend to 0. Then the above gives

$$1=\operatorname*{Lim}_{h=+0}\left\{\frac{h\,X_1(h\theta)}{X_2(h)}\right\};$$

the other terms in the inequality singly tending to 0.

But X_2 is of the same order as $h\,X_1$; therefore the right side gives a function of θ, say $\phi(\theta)$, whence we get $\theta(+0)$.

(b) The other case may be dealt with by using the results of Arts. 10-13 of my paper in *Crelle's Journal* and it will be found that $\theta(+0)$ is non-existent.

§ 7

Certain general types of $f(h)$ with the necessary and sufficient condition for the existence of $\theta'(+0)$ for each type.

15. *Types I and II.*—As regards Type I, it follows from Art. 13, that $\operatorname*{Lim}_{h=0}\frac{\theta-\frac{1}{2}}{h}=\operatorname*{Lim}_{h=0}\frac{\cos\psi(\frac{h}{2})}{h^2\,\psi'(\frac{h}{2})}$. Therefore $\theta'(+0)$ exists or not according as $\psi(v)\succ\frac{1}{v}$ or not; when $\theta'(+0)$ exists it equals 0.

Similarly, as regards Type II, it follows from Art. 14, that, if $\frac{\chi}{\psi'} \prec h$, $\theta'(+0)$ exists or not according as $\frac{\chi(v)}{X_2(v)\,\psi'(v)} \prec 1$ or not.

Example. $\chi(v)=v^{-\frac{1}{2}}$, $\psi(v)=\frac{1}{v^m}$. $\theta'(+0)$ exists according as $m>1$ or not. If $m>1$, $\theta'(+0)=0$.

16. *Type III:* $w(t)=\int_0^t W(v)\,dv$, $W(v)=2+\int_0^v Y(u)\,du$, $Y(u)=2+\sin\psi(u)$.

$\theta'(+0)$ exists or not according as

$$\psi \succ \log\frac{1}{u}$$

or

$$\psi \preceq \log\frac{1}{u}.$$

Proof:

(*a*) Let $\psi \succ \log\frac{1}{u}$. Then

$$w(v)=2+2v-\frac{\cos\psi(v)}{\psi'(v)}+0_{23}(v),$$

$$w(t)=2t+t^2-\frac{\sin\psi(t)}{\{\psi'(t)\}^2}+0_{24}(t^2),$$

$$f(h)=h^2+\frac{h^3}{3}+\frac{\cos\psi(h)}{\{\psi'(h)\}^3}+0_{25}(h^3).$$

The above is, by (M_1), $=h\,w(\xi)=h\left[2\xi+\xi^2-\frac{\sin\psi(\xi)}{\{\psi'(\xi)\}^2}+0_{24}(\xi^2)\right]$.

Dividing both the sides by h^2, we have

$$1+\frac{h}{3}+\frac{\cos\psi(h)}{h^2\{\psi'(h)\}^3}+0_{26}(h)=2\theta+h\theta^2-\frac{\sin\psi(h\theta)}{h\{\psi'(h\theta)^2\}}+0_{27}(h).$$

Now $\psi'(h) \succ \frac{1}{h}$; therefore $\frac{1}{h^2\{\psi'(h)\}^3} \prec h$, $\frac{1}{h\{\psi'(h\theta)\}^2} \prec h$. Thus,

$$2\theta = 1 + \frac{h}{3} + 0_{28}(h) + h\theta^2.$$

Hence $\theta(+0)$ exists and equals $\frac{1}{2}$, and $\theta'(0)$ exists being $\frac{1}{24}$.

(b) Let $\psi \precsim \log\frac{1}{u}$. Then, proceeding as in (b) of Art. 13, we have

$$h^2 + \frac{h^3}{3} + A_2 h^3 \cos\{\psi(h) + B_2\} + 0_{29}(h^2) = hw(\xi) = h^2\{2\theta + h\theta^2 + A_1 h\theta^2 \cos\{\psi(h\theta) + B_1\} + 0_{30}(h)].$$

Dividing both the sides by h^2, we have

$$1 + \frac{h}{3} + A_2 h \cos\left\{\psi(h) + B_2\right\} = 2\theta + h\theta^2 + A_1 h\theta^2 \cos\{\psi(h\theta + B_1)\} + 0_{31}(h).$$

Thus $\theta(+0)$ exists and equals $\frac{1}{2}$;

but for h small $\dfrac{\theta - \frac{1}{2}}{h} = \dfrac{\frac{1}{12} + \frac{A_2}{3}\cos\{\psi(h) + B_2\} - \frac{A_1}{4}\cos\{\psi(\frac{h}{2}) + B_1\}}{2 + \frac{h}{2} + A_1 \times \frac{h}{4}\cos\{\psi(\frac{h}{2}) + B_1\}}$

which cannot tend to any limit as h tends to 0.

17. *Type IV:* $w(t) = \int_0^t W(v)\,dv$, $W(v) = 2 + \int_0^v (Y(u)du$,

$$Y(u) = \chi(u)\{2 + \sin\psi(u)\},\ \chi \succ 1.$$

$\theta'(+0)$ exists or not according as

$$\psi \succ \log\frac{1}{u}$$

or

$$\psi \precsim \log\frac{1}{u}$$

Proof:

(*a*) Let $\dfrac{\chi(u)}{\psi'(u)} \prec u$. Then, denoting $\int_0^h X_2(t)\,dt$ by $X_3(h)$,

$$W(v)=2+2X_1(v)-\frac{\chi(v)}{\psi'(v)}\cos\psi(v)+0_{21}(v),$$

$$w(t)=2t+2X_2(t)-\frac{\chi(t)}{\{\psi'(t)\}^2}\sin\psi(t)+0_{22}(t^2),$$

$$f(h)=h^2+2X_3(h)+\frac{\chi(h)}{\{\psi'(h)\}^3}\cos\psi(h)+0_{23}(h^3).$$

$$=h\{2\xi+2X_2(\xi)-\frac{\chi(\xi)}{\{\psi'(\xi)\}^2}\sin\psi(\xi)+0_{22}(\xi^2)\} \text{ by } (M_1).$$

Dividing both the sides of the above equation by h^2, we have

$$2\theta=1+\frac{2X_3(h)}{h^2}+\frac{\chi(h)}{h^2\{\psi'(h)\}^3}\cos\psi(h)+0_{24}(h)$$

$$-\frac{2X_2(h\theta)}{h}+\frac{\chi(h\theta)}{h\{\psi'(h\theta)\}^2}\cos\psi(h\theta)+0_{25}(h). \qquad \dots \quad (1)$$

Now $X_3(h)$ is obviously $\prec h^2$, $\dfrac{\chi}{h^2(\psi')^3}\prec h$, $\dfrac{X_2}{h}\prec 1$, $\dfrac{\chi}{h(\psi')^2}\prec h$.

Therefore (1) gives $\theta(+0)=\frac{1}{2}$. Hence

$$\frac{\theta-\frac{1}{2}}{h}=\frac{X_3(h)}{h^3}+\frac{\chi(h)}{2h^3\{\psi'(h)\}^3}\cos\psi(h)-\frac{X_2(h/2)}{h^2}+\frac{\chi(h/2)}{h^2\{\psi'(h/2)\}^2}\cos\psi(h/2)$$

$$+0_{26}(1) \qquad \dots \quad (2)$$

Now $\dfrac{X_3}{h^3}$ and $\dfrac{X_2}{h^2}$ are of the same order as χ; also

$$\frac{\chi}{h^3(\psi')^3} \text{ and } \frac{\chi(\frac{h}{2})}{h^2\{\psi'(\frac{h}{2})\}^2}$$

are both $\prec\chi$. Therefore $\displaystyle\lim_{h=+0}\frac{\theta-\frac{1}{2}}{h}$ exists and is infinite.

(*b*) The other case may be dealt with as in (*b*) of Art. 14. For example, take $\psi=\log\dfrac{1}{u}$. Then it is easily proved by using the result of Art. 12 of my paper in *Crelle's Journal* that $\theta'(+0)$ is non-existent.

§ 8

Functions θ each non-differentiable at the points of an everywhere dense set.

18. *Functions θ non-differentiable at the rational points: Type A.*

(α) Let $\phi(z)$ stand for $10z^2+z^2 \cos\log\frac{1}{z^2}$. Then $\phi'(z)$ will be a monotone, increasing and continuous function for every value of z, positive or negative.

$$\text{For, } \phi'(z)=20z+2\sqrt{2}z \cos\left(\log\frac{1}{z^2}-\frac{\pi}{4}\right),$$

$$\phi''(z)=20+2\sqrt{10}\cos\left(\log\frac{1}{z^2}-\frac{\pi}{4}-\tan^{-1}\tfrac{1}{2}\right).$$

Therefore it is proved that, for every value of z, $\phi''(z)>0$ and therefore $\phi'(z)$ is monotone and increasing; that it is also continuous is obvious.

(β) Thus, if $\{\omega_n\}$ be the set of rational points in the interval $(0,1)$,

$$f(h)=\sum_{n=1}^{\infty}\frac{\phi(h-\omega_n)-\phi(\omega_n)}{n^2}$$

will give $f(0)=0$ and $f'(h)=\sum_{n=1}^{\infty}\frac{\phi'(h-\omega_n)}{n^2}$ which will be monotone, increasing, and continuous for every value of h in $(0, 1)$, because such is $\phi'(h-w_n)$.

Therefore, by Theorem 2, there is one-to-one correspondence between h and ξ as each moves in its domain of variability. If, then, ξ is a rational number, say ω_m, there will be a corresponding value of h, say h_m. Now, by (M_1),

$$f(h)=hw(\xi).$$

Since $f'(h)$ exists for every value of h,

$$f'(h_m)=\left[\frac{d}{dh}\{hw(\xi)\}\right]_{h=h_m}=w(\xi_m)+h_m\left\{\frac{d}{dh}w(\xi)\right\}_{h=h_m}.$$

Thus it is proved that, for $h=h_m>0$,

$$\frac{d}{dh}\{w(\xi)\} \text{ exists and equals } \frac{f'(h_m)-w(\xi_m)}{h_m}.$$

But $\left(\frac{d\xi}{dh}\right)_{h=h_m}$ must not exist; for, if it were to exist, $w'(\xi_m)$ would exist and, in fact, equal $\dfrac{\left[\frac{d}{dh}\{w(\xi)\}\right]_{h=h_m}}{\left(\frac{d\xi}{dh}\right)_{h=h_m}}$, which will be absurd as $w'(\xi_m)$ is non-existent.

If 0 is included in $\{\omega_n\}$, it can be proved that $\theta(+0)$ is non-existent because of the inclusion of the term $h^2 \cos \log \frac{1}{h^2}$ in $f(h)$.

19. *A Broden's function taken for* $w(t)$. *Type* $(B.)$—The three non-differentiable functions given by Broden* in *Crelle's Journal*, Bd. 118, are all monotone, increasing and continuous. Therefore, if $f(h)=\int_0^h w(t)dt$, where $w(t)$ is one of such functions, there will be one-to-one correspondence between h and ξ as each moves in its domain of variability. Now each of the functions is non-differentiable at the points of an everywhere dense set, enumerable or

* Denoting in each case the function by $f(x)$ for $(0,1)$ where the ends are primary points, Broden says the following about the three functions: (1) "The function is continuous and throughout increases with x; the derivatives $f'+(x)$ and $f'-(x)$ are everywhere definite, finite and different from 0; they are also equal to one another (and to $f'(x)$) with the exception of an enumerable and everywhere dense set of x-values (*viz.*, those corresponding to the primary points). (See pp. 27 and 28 of Broden's paper.) (2) "$f(x)$ is continuous and throughout increases with x; for a certain enumerable and everywhere dense set of x-values (*viz.*, those corresponding to the primary points) $f'_-(x)=0$, and $f'_+(x)$ has a definite value >0; for an unenumerable and everywhere dense set of x-values is $f'_-(x)=f'_+(x)=0$; for another such set is $f'_-(x)$ non-existent but $f'_+(x)$ existent and >0. (See p. 37 *l.c.*) (3) "$f(x)$ is continuous and increases throughout with x; for an everywhere dense and enumerable set of x-values is the regressive derivative $f'_-(x)=0$ but the progressive $f'_+(x)=\infty$; for an everywhere dense and unenumerable set of x-values is $f'_-(x)=f'_+(x)>0$; for a second such set is $f'_-(x)=f'_+(x)=0$; for a third $f'_-(x)=f'_+(x)=\infty$; for a fourth set of the same kind are $f'_-(x)$ and $f'_+(x)$ both non-existent; for a fifth is $f'_-(x)=0$ but $f'_+(x)$ non-existent." (See pp. 46 and 47, *l.c.*) The first set in the above includes the primary points.

unenumerable. Thus, there are, corresponding to these functions of Broden, functions θ each of which is non-differentiable at an everywhere dense set, enumerable or unenumerable.

An interesting question arises : As in the case of each function of Broden, 0 is a primary point and, therefore, a point of non-differentiability for $w(t)$, $w'(0)$ is non-existent ; what can be said about the existence of $\theta(+0)$ and $\theta'(+0)$?

The answer to this question may be attempted as follows :

(*a*) For each of the first two functions $w'(+0)$, which has been designated $f''(0)$ in the preceding pages, exists, is finite and greater than 0 ; hence in the case of each, by Dini's result, $\theta(+0)$ exists and equals $\frac{1}{2}$. In the case of the third function of Broden, $w'(+0)$ is ∞, and it is not difficult to prove that $\theta(+0)$ exists.

(*b*) A regards $\theta'(+0)$, its existence is unlikely ; for, in any neighbourhood of 0 ever so small the points where $w'(h)$ is non-differentiable are everywhere dense.

§ 9.

Concluding Remarks.

20. The results obtained in the preceding articles enable us to attempt an answer to the question :* Is it true that, corresponding to every prescribed function $\theta(h)$, there exists a function $f(h)$ for which (M_1),

$$viz., \quad f(h) = hf'(h\theta),\ 0 < h < 1,$$

holds, and if such a function $f(h)$ does not always exist what conditions must be satisfied by the prescribed function $\theta(h)$ in order that $f(h)$ may exist ?

(*a*) First, let us *assume* that $\theta(h)$ is expansible in the form

$$\theta(h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \ldots \text{to infinity},$$

* A similar question for θ as a function of x and h has been discussed by Rothe in his interesting paper. (See pp. 315-323, *l.c.*) But the treatment breaks down when one tries to apply it to the case when x is kept fixed. Moreover, from the very beginning of his discussion, Rothe postulates the existence of the differential co-efficients of θ of the first three orders. So the conditions given in (*b*) below are *much less restrictive* than those given by Rothe in his discussion of $\theta(x, h)$. Hayashi's investigation of the expansion of $\theta(h)$ in powers of h, when $f(h)$ is given, may be said to have been anticipated in 1880 by Whitcomb, who published in the *American J. M., Vol. 3*, his method for finding the expansion and actually gave the first six co-efficients, including the constant $\frac{1}{2}$; both Rothe and Hayashi seem to be ignorant of Whitcomb's work. B. N. Pal has recently given two more co-efficients. (See *Bulletin, Calcutta M.S.*, Vol. 19.)

and also confine our search for $f(h)$ *to only such functions* as are expansible in the form

$$f(h)=A_0+A_1h+A_2h^2+A_3h^3+\ldots\text{to infinity.}$$

Now, without any loss of generality, take A_0 and A_1 to be zero. Then, by (M_1),

$$\sum_2^{\infty} A_n h^n = h\left\{\sum_2^{\infty} nA_n(h\theta)^{n-1}\right\}$$

$$=2h^2A_2\sum_0^{\infty} a_n h^n+3A_3h^3\left(\sum_0^{\infty} a_n h^n\right)^2+4A_4h^4\left(\sum_0^{\infty} a_n h^n\right)^3$$

$$+5A_5h^5\left(\sum_2^{\infty} a_n h^n\right)^4+\ldots\text{to infinity.}$$

By equating the co-efficients of the powers of h in the above to those in $\sum_2^{\infty} A_n h^n$, we get the values of the A's, *provided that a_0 has its value restricted to numbers of the form* $\left(\frac{1}{m}\right)^{\frac{1}{m-1}}$. Thus if $A_2\neq 0$, a_0 *must* be $\frac{1}{2}$; if $A_2=0$, but $A_3\neq 0$, a_0 *must* be $\left(\frac{1}{3}\right)^{\frac{1}{2}}$; generally, if

$$A_2=A_3=\ldots A_{m-1}=0 \text{ but } A_m\neq 0,$$

then a_0 *must* be $\left(\frac{1}{m}\right)^{\frac{1}{m-1}}$. With this important condition, the A's are determinate and expressible in terms of the a's; *e.g.*, if $A_2\neq 0$,

$$A_3=8a_1A_2,\ A_4=4(a_2+12a_1^2)A_2,\ A_5=2A_2\{a_3+12a_1^3+24a_0a_1a_2+$$

$$24a^2{}_0a_1a_2+24\cdot 12a_0^2a_1^3\}\times\frac{16}{11}$$

and so on for the other A's.

(*b*) The general restrictions on the properties of $\theta(h)$ are (*i*) that, for $h\geq 0$, it must be continuous and (*ii*) that it must *not* be everywhere non-differentiable.

It is thus clear that, in order that $f(h)$ may exist corresponding to $\theta(h)$, $\theta(h)$, in addition to being between 0 and 1 for all values of h, must fulfil certain restrictions and cannot be arbitrary. The following examples will illustrate this remark:

Ex. 1. If θ is constant it can have values only of the form $\left(\frac{1}{m}\right)^{m-1}$ where m is a positive integer.

Exs. 2-4. There is no $f(h)$ corresponding to $\theta=\frac{1}{3}+\frac{h}{2}$. For $\theta=\frac{1}{2}+\frac{h}{3}$,

$f(h)=\mathrm{A}_2\left\{h^2+\frac{8}{3}h^3+\frac{16}{3}h^4+\frac{44\times 2}{99}h^5+\ldots\right\}$, A_2 being any constant

different from 0. For $\theta=\frac{1}{\sqrt{3}}+\frac{h}{3}$, $f(h)=\mathrm{A}_3\left\{h^3+\frac{6}{3\sqrt{3}-4}\times h^4+\ldots\right\}$,

A_3 being any constant different from 0.

Ex. 5. If $\theta=\frac{1}{2}-\frac{1}{8}h\cos\frac{2}{h}$, for small values of h; then for such values $f(h)=h^2-h^4\sin\frac{1}{h}$.

Ex. 6. If $\theta^{\frac{1}{2}}=\frac{2}{3}-3h^{\frac{1}{3}}\left(\frac{2}{3}\right)^{\frac{5}{3}}\cos\frac{1}{\left(\frac{2}{3}\right)^{\frac{2}{3}}h^{\frac{1}{3}}}$ for small values of h,

then for such values $f(h)=\frac{2}{3}h^{\frac{3}{2}}-9h^{\frac{13}{6}}\sin\frac{1}{h^{\frac{1}{3}}}$.

Ex. 7. If $\theta=\dfrac{1+\frac{1}{\sqrt{5}}\cos\left\{\log\frac{1}{h}+\tan^{-1}\frac{1}{2}\right\}}{2+\cos\left\{\log\frac{1}{h}+\frac{1}{\theta}\right\}}$,

then $f(h)$ will be of the form $h^2+\mathrm{C}h^2\cos\left\{\log\frac{1}{h}+\mathrm{D}\right\}$, C and D being suitable constants, and in fact $\mathrm{C}=\frac{1}{\sqrt{5}}$, $\mathrm{D}=\tan^{-1}\frac{1}{2}$.

21. Finally, there are two facts which deserve to be mentioned here, *viz.*, (1) that Sokolowski's typical result relating to $\theta(+0)$ when $f''(0)$ is infinite follows from (*a*) of Art. 14, and (2) that the points where θ is non-differentiable can form only a set of measure 0.

I proceed to prove these two statements one after the other.

(1) Sokolowski's result relating to $\theta(+0)$ when $f''(0)$ is infinite is typified by the following: If a number $\rho > -1$ and $\neq 0$ exists such that $\lim\limits_{h=+0} \dfrac{f'(h)-l}{h^\rho}$ is finite and $\neq 0$, l being a conveniently chosen number, then $\theta(+0)$ exists and equals $(1+\rho)^{-\frac{1}{\rho}}$.

Now, remembering that the procedure of (*a*) of Art. 14 remains valid even if

$$w(t) = \int_0^t \chi(v)\,dv,$$

it is obvious that Sokolowski's result follows. If χ is not integrable in an interval containing 0, $w(t)$ may be taken to be not

$$\int_0^t \chi(v)\,dv \quad \text{but} \quad \int_\alpha^t \chi(v)\,dv$$

where $\alpha \gtrless 0$. Denoting by $X_1(t)$ the latter integral the formula

$$1 = \lim_{h=+0} \left\{ \frac{hX_1(h\theta)}{X_2(h)} \right\}$$

holds.

Examples: $\chi(v) = v^{-\frac{1}{3}}$; $\chi(v) = v^{-\frac{1}{2}}$; $\chi(v) = \log \dfrac{1}{v}$.

(2) The second statement, that $\theta(h)$ can be non-differentiable at the utmost at the points of a set of measure 0, may be proved as follows:—

From the equation

$$\frac{f'(\mathrm{H}_1)-w(s_1)}{\mathrm{H}_1}=w'(s_1)\times\left(\frac{d\xi}{dh}\right)_{h=\mathrm{H}_1}$$

of Art. 3, p. 160, it is clear that if $\theta(h)$ and, consequently, $\xi(h)$ were non-differentiable at the points of a set of measure greater than 0, the same would hold for $w(t)$. But $w(t)$ is monotone, and therefore the set of points, where it is non-differentiable, can have at the utmost measure zero.

The possibility of a nowhere differentiable θ will be studied in a subsequent paper on the functional properties of $\theta(h)$ as a *multiple-valued* function of h.

SUR UN DÉVELOPPEMENT DES FONCTIONS ABSTRAITES CONTINUES

PAR

MAURICE FRÉCHET (*Paris*)

(*Read December 29, 1928*)

1. *Introduction.*—On sait, depuis Weierstrass, que toute fonction continue $X=F(x)$ peut être représentée comme limite d'une suite de polynomes $P_n(x)$, (la convergence étant uniforme dans tout intervalle fini).

Nous avons pu précédemment étendre ce théorème au cas où X restant un nombre, la variable x désigne une fonction $x(t)$. Nous avons ainsi traité : 1° *le cas où $x(t)$ est une fonction continue* de t * et 2° *le cas où $x(t)$ est une fonction sommable.* † Nous avons pu montrer que $F(x)$ est la limite d'une suite de fonctionnelles $P_n(x)$ dites d'ordres entiers et qui sont la généralisation directe des polynomes. Dans le cas où $x(t)$ est continue, nous avons pris soin de montrer que la généralisation était complète, en donnant une expression précise des $P_n(x)$ et en énonçant les conditions de convergence uniforme.

Nous allons maintenant passer à *un cas beaucoup plus général,* où, pour cette raison même, le résultat obtenu ne sera plus aussi étroitement calqué sur le théorème de Weierstrass.

D'une part, x ne sera plus assujetti à être une fonction $x(t)$, d'autre part, la fonction F, elle même, ne sera pas nécessairement un nombre. De sorte que nous pourrons envisager la relation $X=F(x)$ comme définissant

* " Sur les fonctionelles continues," *Ann. Ec. Norm.* (3) XXVII ; 1910, p. 213.

† " Les fonctions d'une infinité de variables," *C. R. du Congrès Soc. Savantes*, 1909.

la transformation d'un "point" x d'un espace abstrait en un "point" X d'un espace abstrait distinct ou non du premier.

2. *Les espaces envisagés.*—Nous allons essayer d'approcher de toute transformation continue $F(x)$ au moyen de transformations "d'ordres entiers." Dans un mémoire précédent,* nous avons défini et étudié ces transformations. *Une transformation $X=F(x)$ est d'ordre n lorsqu'elle est continue et telle que sa différence $\Delta^{(n+1)}F(x)$ d'ordre $n+1$ est identiquement nulle, sans qu'il en soit de même de sa différence d'ordre n.* Nous posons ici

$$(1) \qquad \Delta^{(n+1)} F(x)=\Delta_{n+1} \Delta_n \ldots\ldots \Delta_2 \Delta_1 F(x) \; ; \quad \text{avec, en général,}$$

$$(2) \qquad \Delta_r \Phi(x)=\Phi(x+\Delta_r x)-\Phi(x)$$

de sorte que $\Delta^{(n+1)} F(x)$ depend de $n+1$ *accroissements* de x qui sont *indépendants* (et non nécessairement égaux).

Pour que cette définition ait un sens, il faut que les espaces où se meuvent x et X satisfassent à certaines conditions.

Mais, pour généraliser le théorème de Weierstrass, il nous sera utile d'introduire de nouvelles conditions. Nous ne nous préoccuperons pas ici de savoir si le résultat obtenu ne pourrait pas subsister dans une catégorie d'espaces plus étendue que la catégorie T á laquelle nous allons nous limiter dans ce qui va suivre. Il nous suffira d'observer : que cette catégorie comprend les espaces particuliers envisagés dans les deux mémoires cités plus haut; que cette catégorie comprend d' autres espaces; et qu'enfin, elle comprend un grand nombre des espaces fonctionnels les plus fréquemment utilisés par les mathematiciens. Nous en indiquerons deux exemples pour terminer.

Nous nous limiterons donc dans la suite à *la catégorie* T *d'espaces que nous allons définir.* Ce sont certains des espaces *(D)affines* † à savoir ceux qui satisfont aux conditions supplémentaires suivantes :

1° Les distances sont inaltérées dans chaque translation.

2° A chaque "point" x est associée une suite (finie ou infinie) de nombres $x_0, x_1, x_2, \ldots\ldots x_n \ldots\ldots$ qu'on peut appeler les coordonnées du point.

3° Ces coordonnées étant des fonctions numériques du point abstrait x

$$(3) \quad x_0=a_0(x), \; x_1=a_1(x), \ldots\ldots x_n=a_n(x), \ldots\ldots,$$

on suppose que ces fonctions sont du premier ordre au sens indiqué plus haut.

* "Les polynomes abstraits," *Journal de Math*, 1929. t. 8.

† Voir la définition, p. 144, de notre livre "Les espaces abstraits" chez Gauthier-Villars, Paris, 1928.

4° On suppose qu'il existe des points abstraits e_i^k tels que pour tout point x de coordonnées $x_1, x_2, \ldots\ldots$, on ait

$$(4) \qquad x = \lim_{n \to \infty} \left(x_0 . e_0^n + \ldots\ldots + x_{r_n} . e_{r_n}^n \right)$$

Il faut observer qu'on peut toujours supposer $r_n = n$, soit, si $r_n < n$, en prenant $e_{r_n+1}^n = \ldots\ldots = e_n^n = 0$, soit, si $r_n > n$, en répétant plusieurs fois la même parenthèse, dans la suite qui tend vers x, depuis le rang n jusqu'au rang r_n. Il est donc équivalent de supposer

$$(5) \qquad x = \lim_{n \to \infty} \left(x_0 . e_0^r + \ldots\ldots + x_r . e_r^r \right).$$

D'après ce qui précède, chaque point x détermine une suite de coordonnées $x_1, x_2, \ldots\ldots$ et si une suite de coordonnées $x_1, x_2, \ldots\ldots$ est donnée, la formule (5) détermine le point x correspondant. Mais nous ne supposons pas que toute suite de nombres peut être considérée comme la suite des coordonnées d'un même point.

Il nous sera utile pour la suite de signaler une conséquence des conditions imposées aux espaces considérés ; pour une valeur de r déterminée, le point abstrait

$$(6) \qquad x^{(r)} = x_0 . e_0 + \ldots\ldots + x_r . e_r$$

où $e_0, \ldots\ldots e_r$ sont des points abstraits fixes, dépend continuement du système de nombres $x_0 \ldots . x_r$. Pour le montrer, observons d'abord que la transformation $x^{(2)}$ dépend continuement du système de nombres x_0, x_1. En effet, d'après la condition 1° et en désignant par $[x, y]$ la distance de deux points abstraits x, y, on aura

$$(7) \qquad [x_0 . e_0 + x_1 . e_1, x_0' . e_0 + x_1' . e_1] = [(x_0 - x_0') . e_0, (x_1' - x_1) . e_1]$$
$$\leq [0, (x_0 - x_0') . e_0] + [0, (x_1' - x_1) . e_1].$$

Quand x_0, x_0', x_1, x_1' varient, les points $(x_0 - x_0') . e_0$ et $(x_1' - x_1) . e_1$ se déplacent respectivement sur deux " droites " fixes passant par l'origine. Les "*longueurs*" des vecteurs allont de l'origine à ces deux points sont $| x_0 - x_0' | \times \| e_0 \|$, $| x_1' - x_1 | \times \| e_1 \|$, elles sont petites quand x_0' et x_1' sont voisins de x_0 et x_1. Or, l'espace étant un espace (D) affine, il en résulte

que les *distances* correspondant à ces deux longueurs sont petites en même temps. Par suite, le premier membre de (7) est aussi petit, ce qu'il s'agissait d'établir.

De même, on a

$$[x_0 . e_0 + x_1 . e_1 + x_2 . e_2, \; x_0' . e_0 + x_1' . e_1 + x_2' . e_2]$$

$$= [(x_0 - x_0') . e_0 + (x_1 - x_1') . e_1, \; (x_2' - x_2) . e_2]$$

$$\leq [0, (x_0 - x_0') . e_0 + (x_1 - x_1') . e_1] + [0, (x_2' - x_2) . e_2]$$

$$\leq [0, (x_0 - x_0') . e_0] + [0, (x_1' - x_1) . e_1] + [0, (x_2' - x_2) . e_2].$$

On voit de la même façon que le premier membre tend vers zéro quand x_0', x_1', x_2' tendent vers x_0, x_1, x_2. Et ainsi de suite jusqu'à $x^{(r)}$.

3. *Démonstration.*—Soit, maintenant, $X = F(x)$, une relation entre deux points x et X appartenant à deux espaces (distincts ou non) de la catégorie T spécifiée plus haut. On a donc par hypothèse

$$(8) \qquad x = \lim_{r \to \infty} [x_0 . e_0^r + \ldots \ldots \ldots + x_r . e_r^r]$$

$$(9) \qquad X = \lim_{s \to \infty} [X_0 E_0^s + \ldots \ldots + X_s . E_s^s]$$

Supposons maintenant que la transformation $X = F(x)$ soit continue; comme la coordonnée X_k de X est une fonction continue $A_k(X)$ du point X, il en résulte, en vertu de la continuité des fonctions abstraites continues composées (E. A., p. 240), que la fonction $X_k = A_k(F(x))$ est une fonction continue de x. Mais alors en vertu de (8), on aura

$$(10) \qquad X_k = \lim_{r \to \infty} A_k (F(x_0 . e_0^r + \ldots \ldots + x_r . e_r^r)) \qquad \text{ou}$$

$$(11) \qquad X_k = \lim_{r \to \infty} f_{k, r} (x_0, \ldots \ldots x_r)$$

D'où, en vertu de (9)

(12) X=

$$\lim_{s\to\infty}\left(\lim_{r\to\infty}\left[f_{0,r}(x_0,\ldots x_r)\ E_0^s + \ldots\ldots f_{s,r}(x_0,\ldots x_r).\ E_s^s\right)\right]$$

Ici, $f_{k,r}(x_0,\ldots\ldots x_r)$ est une fonction numérique des nombres $x_0,\ldots\ldots x_r$ et cette fonction est, comme $x_0 . e_0^r + \ldots\ldots + x_r . e_r^r$, nécessairement continue par rapport à l'ensemble des $x_0,\ldots\ldots x_r$. Par conséquent d'après le théorème primitif de Weierstrass, $f_{k,r}$ est la limite d'une suite de polynomes $Q_{k,r,h}$

$$(13)\quad f_{k,r}\ (x_0,\ldots\ldots x_r) = \lim_{h\to\infty} Q_{k,r,h}\ (x_0,\ldots\ldots x_r).$$

D'où, en portant dans le crochet de (12)

$$(14)\quad f_{0,r}\ .\ E_0^s + \ldots\ldots + f_{s,r}\ .\ E_s^s$$
$$= \lim_{h\to\infty}\left[Q_{0,r,h}\ .\ E_0^s + \ldots\ldots + Q_{s,r,h}\ .\ E_s^s\right]$$

Mais puisque les $Q_{k,r,h}$ sont des polynomes en $x_0,\ldots\ldots x_r$ le crochet de (14) est un point abstrait représentable sous la forme

$$(15)\quad P_{k,r,s}\ (x_0,\ldots\ldots, x_r) = \sum_{\alpha_0,\ldots\alpha_r} x_0^{\alpha_0}\ \ldots\ldots x_r^{\alpha_r}\ .\ M_{h,r,s}^{(\alpha_0,\ldots\ldots\alpha_r)}$$

où les M sont des points abstraits. Et l'on a

$$(16)\quad X = F(x) = \lim_{s\to\infty}\left(\lim_{r\to\infty}\left[\lim_{h\to\infty}\left\{P_{h,r,s}\ (x_0,\ldots x_r)\right\}\right.\right..$$

Nous avons donc déjà établi que toute transformation abstraite continue peut être représentée comme triple limite de transformations abstraites qui sont des polynomes *par rapport aux coordonnées* du point transformé x.

Mais nous pouvons aller plus loin. Utilisons les formules (3) et posons

$$(17)\qquad P_{h,r,s}(x) = P_{h,r,s}\ (a_0(x),\ldots\ldots a_r(x))$$

Nous allons montrer que $Z = P_{h,r,s}\ (x)$ est une transformation abstraite *d'ordre entier* du point x dans un point Z appartenant, comme les M, aux même espace que celui où se déplaçait X.

D'abord, en vertu de la continuité des fonctions abstraites continues composées, $P_{h,r,s}\ (x_0, x_1\ldots\ldots x_r)$, $a_0\ (x),\ldots\ldots a_r\ (x)$, la fonction $P_{h,r,s}\ (x)$ est

continue. D'autre part, soit n, la plus grande valeur de la somme des degrés $a_0+a_1+\ldots\ldots a_r$ dans (15). Si on donne à x des accroissements indépendants $\Delta_1 x$, $\Delta_2 x, \ldots \Delta_{n+1} x$, chaque coordonnée x_k recevra des accroissements $\Delta_1 x_k$, $\Delta_2 x_k, \ldots$, $\Delta_{n+1} x_k$. Si $x_k=a_k(x)$ était une fonction quelconque de x, à $x+\Delta_1 x+\Delta_2 x$, par exemple, correspondrait pour x_k. $x_k+\Delta_1 x_k+\Delta_2 x_k + \Delta_2 \Delta_1 x_k$. Mais $a_k(x)$ était du premier ordre, on a par definition $\Delta_2 \Delta_1 a_k(x)=0$, donc, ici, à $x+\Delta_1 x+\Delta_2 x$ correspond $x_k+\Delta_1 x_k+\Delta_2 x_k$. Et de même pour les autres combinaisons analogues de x, $\Delta_1 x, \ldots \Delta_{n+1} x$.

Il en résulte que $\Delta^{(n+1)} P_{k,r,s}(x)=\Delta_{n+1} \Delta_n \ldots\ldots \Delta_1 P_{k,r,s}(x)$ correspondant aux accroissements indépendants $\Delta_1 x, \ldots\ldots, \Delta_{n+1} x$ s'obtient en calculant la valeur de la différence

$$\Delta^{(n+1)} P_{k,r,s}(x_0, \cdots\cdots x_r)$$

$$=\sum_{a_0,\ldots a_r} \left[\Delta^{(n+1)} x_0^{a_0} \ldots\ldots x_r^{a_r}\right] . M_{k,r,s}^{(a_0 \ldots\ldots a_r)}$$

Dans celle-ci

$$\Delta^{(n+1)} x_0^{a_0} \ldots\ldots x_r^{a_r}=\Delta_{n+1} \Delta_n \ldots\ldots \Delta_1 x_0^{a_0} \ldots . x_r^{a_r}$$

s'obtient en donnant au point $(x_0 \cdots \cdots x_r)$ de l'espace à r dimensions, $n+1$ accroissements $(\Delta_1 x_0, \ldots\ldots \Delta_1 x_r), \ldots\ldots (\Delta_{n+1} x_0, \ldots\ldots \Delta_{n+1} x_r)$ suivant les formules usuelles et comme $a_0+\ldots+a_r \leq n$, cette différence $(n+1)$ième est nulle. Les $\Delta_i x_k$ ne sont peut-être pas indépendants, mais les $\Delta_i x$ le sont et cela suffit pour qu'on puisse conclure que $P_{k,r,s}(x)$ est d'ordre entier. Finalement, nous avons démontré que :

Si les variables abstraites x et X restent dans deux espaces (distincts ou non) de la catégorie T définie au § 2, toute transformation continue $X=F(x)$ de x en X peut être représentée sous la forme

$$(18) \qquad X=F(x)=\lim_{s \to \infty}\left(\lim_{r \to \infty}\left[\lim_{h \to \infty}\left\{P_{k,r,s}(x)\right\}\right]\right)$$

d'une limite triple de transformations d'ordres entiers

$$(19) \qquad X^{(h,r,s)}=P_{k,r,s}(x)$$

4. *Remarque.*—Il serait intéressant de chercher à obtenir un énoncé plus rapproché de celui de Weierstrass, 1° en substituant à la famille des fonctions

à triple indice $P_{n,r,s}$ une suite de fonctions à un seul indice, (c'est à dire en remplaçant la triple limite par une simple limite) ; 2° en précisant les conditions de convergence unforme. Les deux questions sont d'ailleurs liées. Comme nous l'avons vu dans le cas plus simple que nous avons autrefois envisagé, elles conduiraient à l'introduction des ensembles compacts et à une démonstration plus délicate et plus longue. Mais le principe de la démonstration que nous venons d'exposer serait encore à la base du nouveau raisonnement. Pour ne pas allonger le présent mémoire, nous nous contenterons donc ici du résultat obtenu et des indications qui précédent. Nous détaillerons celle-ci dans une autre publication.

Toutefois, il nous parait nécessaire de montrer que nous venons bien d'obtenir un résultat nouveau, et pour cela de citer au moins un exemple d'un espace fonctionnel, important pour les mathématiciens, qui rentre dans la catégorie T sans se confondre avec les deux espaces particuliers précédemment traités.

5. *Un exemple d'espaces fonctionnels où le développement précédent est possible. L'espace I des fonctions entières.*—Soit, (I), l'espace (D) affine défini dans notre livre, page 145. Il est clair que la condition 1° du paragraphe précédent est vérifiée. En développant un élément de l'espace I—c'est à dire une fonction entière $x(z)$—en série suivant les puisances de z.

$$x(z)=x_0+x_1 . z+\ldots\ldots+x_n . z^n+\cdots$$

on voit que tout élément $x(z)$ détermine une suite de coefficients $x_0,\ x_1,\ldots x_n$ ......qu'on peut appeler ses coordonnées. De plus, on voit que la condition (5) est réalisée en prenant

$$e_0^r=1,\ e_1^r\ =z,\ldots\ldots,\ e_1^r=z^r$$

ce qui montre, en passant, que dans le cas actuel les e_k^r sont indépendants de r.

Enfin, si O est un cercle ayant pour centre l'origine, on sait qu'on a

$$(20)\qquad a_n(x)=x_n=\frac{1}{2\pi i}\int_O \frac{x(\xi)d\xi}{\xi^{n+1}}$$

Ceci montre que $a_n(x)$ est une fonction du premier ordre de l'élément x: D'une part

$$\Delta_2 \Delta_1 a_n(x) = \frac{1}{2\pi i} \int_C \Delta_2 \Delta_1 \, x(\xi) \times \frac{d\xi}{\xi^{n+1}} = 0 \tag{21}$$

puisque les accroissements $\Delta_2 x$, $\Delta_1 x$ étant indépendants, on a $\Delta_2 \Delta_1 x \equiv 0$. D'autre part, il est clair que, si l'élément $x^{(p)}$ tend vers l'élément x, c'est à dire si la fonction $x^{(p)}(\xi)$ converge vers $x(\xi)$, uniformément dans toute aire finie et en particulier sur C, alors $a_n\left(x^{(p)}\right)$ tendra vers $a_n(x)$

Finalement l'espace I appartient bien à la catégorie T.

ZUR THEORIE DER RELATIV-ABELSCHEN KÖRPER

VON

RUDOLF FUETER (ZÜRICH).

(Communicated by Professor Ganesh Prasad)

[*Read July 29, 1928*]

Durch die grundlegende Arbeit von *T. Takagi* im Vol XLI des Journal of the College of science (Tokyo, 1920) hat man die Einsicht erhalten, dass jedem relativ-Abelschen Korper K eines gegebenen Körpers k ein bestimmter Strahl des Grundkörpers zugeordnet ist, und dass umgekehrt zu jedem Strahl von k ein bestimmter Klassenkörper gehört, wobei unter Strahl stets ein Congruenzstrahl modulo einem bestimmten Ideal $\mathfrak{f}$ als Führer zu verstehen ist. Damit ist Ordnung in die manigfaltigen Erscheinungen gebracht worden, und eine merkwürdige Uebereinstimmung zwischen den relativ-Abelschen Körpern und den Strahlen des Grundkörpers festgestellt. Allein die wirkliche Durchführung der Zahlentheorie der relativ-Abelschen Körper enthält noch gewisse Schwierigkeiten, die teils durch die zum Beweise notwendigen Methoden, teils durch die vorhergehenden Vereinfachungen bedingt sind, die deren Anwendung erst gestatten. Ich möchte daher im Folgenden die Gedanken entwickeln, die mir die Ueberwindung derselben zu erlauben scheinen.

Es sei ein zu k relativ-Abelscher Körper K vom Relativ-grad n gegeben. Die erste Aufgabe ist, den Führer $\mathfrak{f}$ des K zugeordneten Strahls in k zu finden. Dazu berechnet man die Relativdifferente von K in Bezug auf k; $\mathfrak{f}$ ist dann durch alle und nur die Primidealen, die in der Relativdiscriminante aufgehen, teilbar. Wir bezeichnen dieselben mit $\mathfrak{l}_1, \mathfrak{l}_2, \ldots\ldots \mathfrak{l}_t$. Ist dann $\mathfrak{l}_t$ zu n teilerfremd, so ist $\mathfrak{f}$ nur durch die erste Potenz von $\mathfrak{l}_t$ teilbar anzunehmen. Ist dagegen $\mathfrak{l}_t$ ein Teiler von n, und ist l_t die durch $\mathfrak{l}_t$ teilbare rationale Primzahl, so bestimmen wir zuerst die Potenz von $\mathfrak{l}_t$ durch die (l_t) in k teilbar ist. Es sei:

$$l_t \equiv 0 \ (\text{mod. } \mathfrak{l}_t^{s_t}), \ \not\equiv 0 \ (\text{mod. } \mathfrak{l}_t^{s_t+1}).$$

Ferner bestimmen wir die relative Verzweigungsgruppe von $\mathfrak{l}_\iota$ in K relativ zu k. Dieselbe muss eine Abelsche Gruppe sein, deren Ordnung eine Potenz von l_ι ist und besitze den Typus:

$$(l_\iota^{v_1}, l_\iota^{v_2}, \ldots\ldots, l_\iota^{v_s}).$$

Es sei $v^{(i)}$: die grösste aller Zahlen v_r. Wir nehmen dann $\mathfrak{f}$ teilbar an durch (siehe Takagi, a. a. O. pg. 95):

$$\mathfrak{l}^{(v^{(i)}+1)s_\iota+1} = \mathfrak{l}_\iota^{u_\iota+1} \quad \text{wo } u_\iota = (v^{(i)}+1)s_\iota \text{ ist.}$$

Zu diesem $\mathfrak{f}$ bestimmen wir ein Ideal $\mathfrak{F}$ von K, das durch alle in $\mathfrak{f}$ aufgehenden Primideale von K teilbar sei, and zwar zur ersten Potenz, wenn dasselbe zu n teilerfremd ist, und zur Potenz ul^w+1, wenn $\mathfrak{l}$ in n aufgeht. Dabei ist $\mathfrak{l}^w$ die Ordnung der Verzweigungsgruppe von $\mathfrak{l}$ in K.

Es ist zu bemerken, dass im allgemeinen $\mathfrak{F}$ resp. $\mathfrak{f}$ nicht dasjenige Ideal mit kleinster Norm ist, das unsern Zwecken dient, und es ist ein wichtiges Problem, dieses $\mathfrak{f}$ mit kleinster Norm zu bestimmen, ein Problem, das merkwürdige Schwierigkeiten bietet. Dazu muss man nur für die $\mathfrak{l}$ die kleinsten Exponenten angeben, die die folgenden Sätze noch erlauben auszusprechen. Die Sätze gelten dann auch ohne weiteres für alle grössern Exponenten.

$\mathfrak{f}$ und $\mathfrak{F}$ stehen nun in dem Zusammenhang:

1. Satz: *Alle Zahlen von* $\mathfrak{F}$ *die in k liegen, bilden das Ideal* $\mathfrak{f}$. Denn da $\mathfrak{f}$ durch $\mathfrak{F}$ teilbar ist, so liegen alle Zahlen von $\mathfrak{f}$ in $\mathfrak{F}$. Nehmen wir umgekehrt eine Zahl von $\mathfrak{F}$, die zugleich in k liegt, so ist sie durch $\mathfrak{L}^{ul^w+1}$ teilbar, falls $\mathfrak{L}$ ein in $\mathfrak{l}$ enthaltenes Primideal von K ist. Also ist sie auch durch $\mathfrak{L}^{(u+1)l^w}$ teilbar. Da dies für jedes in $\mathfrak{l}$ enthaltene Primideal $\mathfrak{L}$ gilt, so muss sie auch durch $\mathfrak{l}^{u+1}$ teilbar sein, also in $\mathfrak{f}$ liegen.

Wir bilden nun in k den Strahl $s(\mathfrak{f})$, and in K den Strahl $S(\mathfrak{F})$. Aus Satz 1 folgt dann ohne weiteres:

2. Satz: *Alle Zahlen von S* $(\mathfrak{F})$, *die in k liegen, bilden den Strahl* $s(\mathfrak{f})$. Ist ferner S irgend eine Substitution der Abel'schen Relativgruppe von K in Bezug auf k, so ist:

$$S\mathfrak{F} = \mathfrak{F} \tag{1}$$

Denn die Relativdifferente von K zu k ist invariant in Bezug auf alle Substitutionen der Relativgruppe. Wenn somit ein Primideal $\mathfrak{L}$ in $\mathfrak{F}$ auftritt, so tritt auch $S\mathfrak{L}$ auf, da $\mathfrak{f}$ alle in der Relativdiscriminante enthaltenen Primideale enthalten soll, Sind aber $\mathfrak{L}'$ and $\mathfrak{L}''$ zwei im selben $\mathfrak{l}$

aufgehende Primideale von K, so besitzen sie denselben Verzweigungskörper, also auch dieselben Exponenten in **F**.

Ist **K** eine Strahlklasse von S(**F**), so enthalte S**K** alle Ideale, die aus denjenigen von **K** durch die Substitution S der Relativgruppe hervorgehen. Wegen (1) ist S**K** wieder eine Strahlklasse von S(**F**).

Wir können daher die Relativnorm von **K** in Bezug auf alle Substitutionen der Relativgruppe von K zu k definieren. Dieselbe ist wieder eine Klasse von S(**F**). Anderseits enthalt sie sicher auch Ideale von k. Sind nun $\mathbf{A}_1$ und $\mathbf{A}_2$ zwei Ideale von **K**, so gibt es eine Strahlzahl α von S(**F**), so dass :

$$\mathbf{A}_1=(\alpha)\mathbf{A}_2, \text{ wo } \alpha\equiv 1 \pmod{\mathbf{F}},$$

Also ist wegen (1) :

$$\mathrm{N}(\mathbf{A}_1)=(\mathrm{N}(\alpha))\mathrm{N}(\mathbf{A}_2),\ \mathrm{N}(\alpha)\equiv 1 \pmod{\mathbf{F}}.$$

Wegen Satz 2 ist N (α)eine Zahl von s(**f**). Somit sind die Normen von $\mathbf{A}_1$ and $\mathbf{A}_2$ in derselben Strahlklasse von s(**f**). Ist **k** diese Klasse, so setzt man :

$$\mathbf{k}=\mathrm{N}(\mathbf{K}).$$

Daraus folgt der :

Satz 3: *Die Normen aller Ideale einer Strahlklasse von S(**F**) liegen in einer Strahlklasse von s(**f**).*

Man nennt daher S(**F**) den *Klassenstrahl* von s(**f**). Satz 3 erlaubt sofort die Definition der Geschlechter, wie sie von mir, *Math. Annalen* 75, pg. 231 u. ff. eingeführt worden ist, anzugeben :

*Definition: Alle Strahlklassen des Klassenstrahls, deren Relativnormen in Bezug auf k in dieselbe Strahlklasse von s(**f**) fallen, bilden ein Geschlecht.*

Die Anzahl der Geschlechter ist endlich, nämlich höchstens so gross, wie die Strahlklassenzahl von s(**f**).

Es gelten nun folgende fundamentalen Satze :

I. *Die Klassenzahl von s(**f**) ist durch den Relativgrad n von K in Bezug auf k teilbar.*

II. *Ist h(**f**) die Strahlklassenzahl von s(**f**), so existieren genau:*

$$\frac{h(\mathfrak{f})}{n}$$

Geschlechter.

III. *Alle Primideale einer Strahlklasse von $s(\mathfrak{f})$ zerfallen in K in derselben Weise.*

Diejenigen Klassen von $s(\mathfrak{f})$, die Relativnormen von Klassen des Klassenstrahls sind, bilden eine Gruppe in Bezug auf die Multiplikation. Dieselbe ist eine Untergruppe der Klassengruppe. Ist $\mathfrak{g}$ die ganze Gruppe, $\bar{\mathfrak{g}}$ diese Untergruppe, so hat die Faktorgruppe $\mathfrak{g}/\bar{\mathfrak{g}}$ die Ordnung n.

IV. *Die Faktorgruppe $\mathfrak{g}/\bar{\mathfrak{g}}$ ist holoedrisch isomorph mit der Relativgruppe von K zu k.*

Wegen III. müssen alle Primideale der Klassen des Strahls $s(\mathfrak{f})$, denen ein Geschlecht entspricht, in Primideale in K zerfallen vom Relativgrade 1.

Von diesen Eigenschaften ist II, die wichtigste. Aus ihr ergeben sich die übrigen nach bekannten Methoden, die schon von *Weber* mit Hilfe der Dirichlet-Dedekind' schen ζ-Funktion angegeben worden sind.

II. ist von mir im Falle, dass k ein quadratisch imaginärer Körper ist, und von *Takagi* für einen allgemeinen Körper k bewiesen worden. Es scheint mir, dass mein dem klassischen Beweise für quadratische Formen nachgebildeter Beweis völlig zum Beweise von II für ein beliebiges k verwendet werden kann. Hiezu macht man zuerst die völlig unwesentliche Vereinfachung, dass man K durch ein System von zu k relativzyklischen Körpern vom Primzahlpotenzrelativgrad auflöst, so dass man n als Potenz der Primzahl l auffassen kann: $n=l^{\lambda}$. Man beweist jetzt successive:

(a) Es gibt in S(F) n von einander verschiedene Strahleinheiten, deren Relativnorm in Bezug auf k eins ist, and die nicht die $(1-S)$. te symbolische Potenz einer Strahleinheit sind. Dabei ist S eine erzeugende Substitution der zyklischen Relativgruppe.

(b) Jede ganze oder gebrochene Zahl von S(F) deren Relativnorm in Bezug auf k eins ist, ist die $(1-S)$. te symbolische Potenz eines Ideals von S(F), Diese Zahlen können nur in d verschiedenen Klassen von S(F) liegen, wo d eine bestimmte Zahl ist.

(c) In jedem Geschlechte sind gleich viele Strahlklassen (mod. F).

(d) Die $h(\mathfrak{f})$ Strahlklassen von $s(\mathfrak{f})$ bilden in S(F) nur $h(\mathfrak{f})/n$ verschiedene Strahlklassen,

(e) Die Zahl aller Klassen von S(F), deren $(1-S)$. te symbolische Potenz in der Hauptklasse liegt, ist höchstens gleich $dh(\mathfrak{f})/n$.

(f) Die Anzahl der Strahlklassen im Hauptgeschlecht sei e. Dann gibt es höchstens e/d Klassen des Hauptgeschlechtes, die $(1-S)$. te symbolische Potenz einer Klasse sind.

(*g*) Ist g die Anzahl der existierenden Geschlechter, so ist daher:

$$ge \leq \frac{e}{d} \cdot \frac{dh(\mathfrak{f})}{n} = e\,\frac{h(\mathfrak{f})}{n},\; g \leq \frac{h(\mathfrak{f})}{n}$$

(*h*) Es können also die Primideale von höchstens $h(\mathfrak{f})/n$ Klassen in Primideale 1. Relativgrades zerfallen. Die Betrachtung der Klassenzahlen, resp, der zugehörigen ζ-Funktionen in K and k zeigen, dass aber auch alle Primideale 1. Grades dieser Klassen zerfallen müssen, dass daher auch alle diese Geschlechter existieren.

Falls $\mathfrak{l}=2$, und unter k und seinen conjugierten reelle Körper auftreten, so ist der engere Aequivalenzbegriff der Ideale zu Grunde zu legen. Siehe über die Durchführung der Beweise auch Kap. VIII. meines Buches: "Vorlesungen über die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen," Teubner, 1927, zweiter Teil. In gewissen Fällen muss die Potenz, zu der $\mathfrak{l}$ in $\mathfrak{f}$ enthalten ist, gegenüber der obigen Definition erhöht werden.

In diesen Sätzen ist, wie Herr *Artin* neulich gezeigt hat, auch das allgemeinste Reziprozitätsgesetz mit enthalten, falls man die *Takagi*-sche Existenz von K für jede Untergruppe $\bar{\mathfrak{g}}$ der Gruppe $\mathfrak{g}$ aller Strahlklassen von $S(\mathfrak{f})$, $\mathfrak{f}$ ein beliebiges Ideal von k, voraussetzt. Siehe *Artin*: Beweis des allgemeinen Reziprozitätsgesetzes, *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, V. Bd. 4. Heft, pg. 353 (1927).

Zürich, den 9, Mai, 1928.

OLIVER HEAVISIDE

By

E. T. Whittaker

[*Read, July 31st, 1928*]

The family of Oliver Heaviside belonged originally to Stockton-on-Tees. His grandfather, Thomas Heaviside, was a builder and auctioneer in that town: and his father, also Thomas, was born there in 1813.

At the age of nineteen the younger Thomas left Stockton to be apprenticed to a wood-engraver in London, where all his working life was spent. His apprenticeship over, he met and married Miss Rachel Elisabeth West, at that time a governess in the Spottiswoode family,[1] but a native of Taunton in Somerset, the daughter of a wine-merchant there. The issue of this marriage consisted of four sons, of whom the youngest, Oliver, the subject of this notice, was born on 18th May, 1850 at 55, King Street, Camden Town.

The family circumstances at the time of Oliver's birth and childhood were very poor. Wood-engraving was not a prosperous calling, and Thomas Heaviside lacked many of the qualities that make for worldly success. The house in King Street was "a horrid place, with a beer shop opposite, a ragged school close by, low neighbours, and bad drains." Here Oliver spent the first thirteen years of his life, "in miserable poverty:" the effect was, in his own words, "disastrous," and "permanently deformed his future life." But from a very early age his mind was set in one direction. "When I was a young child," he wrote many years afterwards,[2] "I conceived the idea of an infinite series of universes, the solar system being an atom in a larger universe on the one hand, and the mundane atom a universe to a smaller atom, and so on."

[1] William Spottiswoode, afterwards President of the Royal Society, may have been one of her pupils.

[2] *Nature*, 28th January, 1904.

Mr. Heaviside was "in debt to the tradespeople most of the time," and to ease the financial burden, his wife set up a school for girls, where it was proposed to begin Oliver's education. The boy rebelled, until his father dragged him across the street to the ragged school, when he accepted the company of the other sex as the lesser evil.

The school failed in 1862, and thereafter lodgings were let. This was more successful, and in 1863 or '4 a better apartment-house was taken at 117, Camden Street, and the circumstances became easier. Oliver was now at a school of which he gave some reminiscences long afterwards in a discussion on the teaching of geometry, in support of his opinion that geometry is essentially an experimental science. "I feel quite certain," he wrote, "that I am right, on this question of the teaching of geometry, having gone through it at school, where I made the closest observations on the effect of Euclid upon the rest of them. It was a sad farce, though conducted by a conscientious, hardworking teacher. Two or three followed, and were made temporarily into conceited logic-choppers, contradicting their parents: the effect upon most of the rest was disheartening and demoralising."

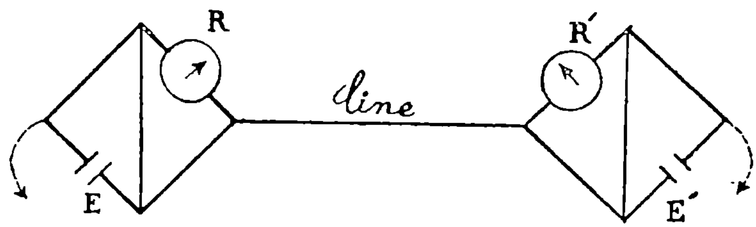
Here also "one day our respected master informed us that it had been found out that water was not HO, as he had taught us before, but something else. It was henceforward to be H_2O_1."

He cannot have got very far in mathematics and chemistry when it became necessary for him to leave school and begin some kind of work. Life at home had not been running smoothly. Thomas Heaviside, a zealous radical and free-thinker, was a man of despotic and unsympathetic temper, who kept his family in order by severe corporal punishment. Of the elder sons, one ran away from home, and one was turned out. Oliver, expecting a similar fate, was glad to hear of an opening away from London which he found by the good offices of a relative who must now be mentioned.

A sister of Mrs. Heaviside's had married well, her husband being Professor Wheatstone of King's College, London, afterwards Sir Charles Wheatstone, F.R.S. Wheatstone had made many improvements in electric telegraphy, and by his influence his nephew Oliver was appointed as a telegraphic operator at the Newcastle office of the Anglo-Danish Telegraph Company, in whose service he remained from 1868 to 1874. He was described as a young man of good appearance, "carefully dressed, with piercing eyes, a fair skin, and light chestnut hair."

At this time telegraphy presented many problems, regarding batteries and resistances in networks of conductors, which required some ingenuity in disposing apparatus and working out the consequences by elementary mathematics. Placed in a situation where such problems originated, and

stimulated by the example of his distinguished relative, Heaviside began to attack them. In 1873 he described some original methods for Duplex telegraphy, *i.e.*, the art of telegraphing simultaneously in opposite directions on the same wire. The general principle underlying these devices was similar to that of the Wheatstone's bridge, *viz.*, that a battery in one arm of the bridge has no effect on a receiving instrument in the opposite arm. Thus if E and E′ are the batteries, R and R′ the receiving apparatus, and if the resistances are suitably adjusted



the battery E produces no current in R and the battery E′ produces no current in R′, so each station only gets the other station's signals. Heaviside then improved the invention so as to make possible quadruplex telegraphy, that is two messages in each direction : the principle was to work a Duplex system with selective apparatus at each end of two kinds, one of which responds to reversals of current, while the other is affected only by the absolute magnitude of the current irrespective of direction.

His early writings do not in general involve more mathematics than algebra, but in 1873 he made use of the Calculus, which he had studied privately in the works of Todhunter: Differential Equations he read in Boole, and Solid Geometry in Gregory and Walton. He was now a regular contributor to the *Telegraphic Journal*, the *English Mechanic*, and the *Philosophical Magazine*, and at the age of 24 had seven papers to his credit.

In 1874, resigning from the service of the Telegraph Co , he returned to live with his parents in London, and henceforth he followed no regular occupation : the reason is not fully known, but may have been connected with his increasing deafness.

An event of the first magnitude in the history of physics had happened just before he left Newcastle, namely, the publication of Maxwell's *Treatise on Electricity and Magnetism*. Heaviside must have mastered it completely within the next two years, for in a paper published in 1876 he made use of the theory of electromagnetic momentum, which is given in Maxwell's second volume. In 1877 another great event took place, the invention of the telephone; and almost at once he began to study the effect of telephonic transmission in causing differences between the sounds produced

at the sending end and those heard in the receiver. In the early days these were very serious, telephony with an underground line was not practicable in 1881 at distances of more than about four miles. Heaviside determined to investigate the matter thoroughly in the light of what he had learnt from Maxwell's treatise, and this was the original motive of his best-known work.

First of all, it was noticed that the telephone was remarkably susceptible to inductive effects: a man walking about a post office telegraphic battery-room with a telephone and a coil of wire could read the Morse signalling from every battery in use. Heaviside therefore began by investigating the electrostatic and electromagnetic induction between two overhead parallel wires. His memoir, which shows a complete mastery of the type of analysis required in higher mathematical physics,—Fourier's theorem and the solution of partial differential equations—is of considerable practical value at the present time, in discussing the interference in telephone circuits due to induction from electric railway currents.

In 1885-87 he contributed to the *Electrician* a series of articles under the title "Electromagnetic Induction and its Propagation," in the first of which Maxwell's theory was presented in what was substantially a new form. Maxwell, following Faraday's ideas, had clearly pointed out that the electric field at each point depended on two vectors, namely, the electric and magnetic forces at the point, and upon the electric and magnetic displacements they produced. But in Maxwell's *Treatise*, the analytical consequences of these principles had not been developed in a straightforward and natural manner: his pages are cumbered with the *débris* of the older theories, with a maze of symbols representing electric and magnetic potentials, vector potentials, and so forth. I well remember, in 1893, buying for myself a second-hand copy of Maxwell which had been the property of a College lecturer on mathematical physics. When I came to the famous chapter on the General Equations of the Electromagnetic field, I found scribbled in his handwriting "from here on this book is absolutely unreadable." The great service which Heaviside now rendered to science was to clear away this accumulation of rubbish, and base the theory on what he called the "duplex" equations

$$\operatorname{curl} H = 4\pi \Gamma$$

$$\operatorname{curl} E = -\dot{B}$$

$$\operatorname{div} \Gamma = 0$$

$$\operatorname{div} B = 0$$

(where H is magnetic force, Γ is electric current, etc.), which modern writers

generally call "Maxwell's equations"—though they are not found in Maxwell's *Treatise*, and the modern writers have in fact copied them from Heaviside.

All writers before Maxwell had regarded electric charge as a kind of substance—the "electric fluid," or, in the two fluid theory, the "electric fluids," vitreous and resinous electricity. Maxwell had introduced a new way of looking at the matter, in which *electric displacement* (present wherever there was electric force) was regarded as the fundamental quantity, and electric charge was regarded merely as the discontinuity of electric displacement at the surface of conductors. Heaviside adopted the Maxwellian view, and expressed it more outspokenly than Maxwell. "Electrification," he wrote, "is nothing more than the divergence of the displacement." He referred contemptuously to "the electricity once supposed to reside upon the surface of conductors," "the imaginary matter of free electricity," and "imaginary accumulations of the electric fluid." In 1888 he modified this position on becoming convinced that cathode rays were streams of moving electrons. But against the tendency to postulate hypothetical substances to account for physical phenomena he strove continually, mocking at "the old fluids, with their absurd and wholly incomprehensible behaviour, their miraculous powers of attracting and repelling one another, of combining together and of separating, and all the rest of that nonsense." "In the development of scientific theories," he wrote, "the further we go, the less we seem to know, by the expurgation of unnecessary hypotheses." "Truth," he said "is even the truest when in its most abstract form."

His views on electricity seemed to be supported by a discovery which he made at this time. Horace Lamb, when studying electric oscillations in spherical and cylindrical conductors in 1883, had found when these conductors are placed in a rapidly-alternating field, the induced currents are almost entirely confined to a superficial layer of the metal. Heaviside now in 1885 generalised this by showing that whatever be the form of the conductor, rapidly-alternating currents do not penetrate far into its substance. In fact, a perfect conductor would be impenetrable to lines of magnetic force: and if the alternations of magnetic force to which a good conductor such as copper are exposed are very rapid, the conductor has not time (so to speak) to display the imperfection of its conductivity, and the magnetic field is therefore unable to extend far below the surface. "The slipping of electrification over the surface of a wire," he said, "is merely the movement of the wave through the dielectric, guided by the wire: and the true mode of establishment of a current in a wire is "the current starting on its boundary, as the result of the initial dielectric wave outside it, followed by diffusion inwards,"

His opinions regarding the nature of electric charge were no doubt the real driving force in the campaign that he waged against what he called "the absurd 4π multiplier" connecting the density of the electric layer on a conductor and the intensity of electric force just outside it, namely,

$$\text{Surface-density} = (4\pi)^{-1} \times \text{force} \times \text{specific capacity},$$

which, he said, was "just as reasonable as it would be to say that, in a conductor,

$$\text{Current} = (4\pi)^{-1} \times \text{E.M.F.} \times \text{conductivity}.$$

He proposed to abolish the 4π's here, by changing the electrical units; defining a unit electric charge as that which produces at distance r an electric force $\frac{1}{4\pi r^2}$: the 4π removed from the previous equation, thus reappears in this last formula. However, as he said "the few formulae where 4π should be, are principally scholastic formulae and little used: the many formulae where it is forced out are, on the contrary, useful formulae of actual practice." "The unnatural suppression of the 4π in the formulae of central force, where it has a right to be, drives it into the blood, there to multiply itself and afterwards break out all over the body of electromagnetic theory."

The proposal was not at first looked on with favour. "When," wrote Fitzgerald, "people tolerate miles and furlongs and packs and bushels and barrels and firkins and hogsheads, etc., etc., how can they be expected to get up any enthusiasm over the eviction of 4π?" However, the new system of units was adopted some years afterwards by Lorentz, since when it has become the fashion.

Another innovation, which led to bitter controversies, was his system of vector analysis. In 1885 he worked out the electromagnetic wave-surface when both the specific inductive capacity and the permeability are aelotropic. "Owing," he says, "to the extraordinary complexity of the investigation, when written out in Cartesian form (which I began doing, but gave up aghast) some abbreviated method of expression becomes desirable......I therefore adopt, with some simplification, the method of vectors, which seems indeed the only proper method......The investigation is thus a Cartesian one modified by certain simple abbreviated modes of expression." This last sentence explains the fundamental difference between Heaviside's vector analysis and Hamilton's quaternions.

Quaternions is not a mere abbreviated mode of expressing Cartesian analysis, but is a branch of mathematics with its own rules of operation, as distinct from ordinary algebra as geometry or mechanics. A quaternion is in fact a "number" in the generalised sense of the word in which, *e.g.*, matrices

are numbers, that is to say, we can assign a meaning to the assertion that two quaternions are *equal*, and we can also define the operations of *adding* and of *multiplying* two quaternions. The multiplication of quaternions however is not commutative (that is, the product qr is not in general equal to the product rq), and quaternions may be defined as that system of numbers, with non-commutative multiplication, for which the vanishing of a product of two factors necessitates the vanishing of one of the factors. The theorem that if $qr=0$ then either $q=0$ or $r=0$ is true only in ordinary algebra (which has commutative multiplication) and in quaternions, which has non-commutative multiplication.

Now the science of Quaternions is dominated entirely by the quaternion and the scalar and vector play a very subordinate part in it, so much so, that the laws of combination of vectors are established by means of quaternions and are made to suit the necessities of quaternions. Thus, since the square of a quadrantal versor is negative (as it must be, since it represents a rotation through 180°), it is necessary that the square of every vector should be a *negative* scalar. Heaviside had studied quaternions in Tait's treatise and had been repelled by this singular convention, "which," he said, "is quaternionically convenient. But in the practical vector analysis of physics it is particularly inconvenient, being, indeed, an obtrusive stumbling-block. All positive scalar products have the *minus* sign prefixed: there is thus a want of harmony with scalar investigations and a difficulty in passing from Cartesians to vectors and conversely." "The inscrutable negativity of the square of a vector in quaternions," he said elsewhere, "is the root of the evil." His own system was a very simple one; it was not a new branch of mathematics, like quaternions, but merely a syncopated form of ordinary Cartesian analysis. The definitions of the scalar product, the vector product, the operator *nabla*, a few transformation-formulae and the integral theorems of Green and Stokes, almost complete the outfit of the vectorist. The orthodox quaternionists Tait and Knott—the "Edinburgh School of Scorners" as Heaviside called them—rushed to the defence of their science, which was attacked not only by Heaviside but at the same time by Willard Gibbs in America. As Knott pointed out, it is impossible to satisfy the associative law if the square of a unit vector is put equal to $+1$; for we should then have

$$i^2.j=j, \qquad \text{while} \qquad i.ij=ik=-ki=-j,$$

so that $(ii)j$ is not equal to $i(ij)$. In 1892-93 the controversy became so violent that Lord Rayleigh was moved to quote Tertullian in a new version, "Behold how these vectorists love one another."

Heaviside was an innovator also in names. "A really practical name," he said, "should be short, preferably monosyllabic. If, in addition,

it be the name or a part of the name of an eminent scientist, so much the better." On these grounds he approved of the *ohm* and the *volt*, and proposed the *mac* (plural max) and the *tom* to keep them company. The *ampére* was not so happy, being a word of two syllables and fearing that it might in practice become *amp* (as indeed has actually happened) he suggested shortening it to *pére*: for, as he said, was not Ampére the father of electrodynamics? Perhaps the time has now come when a unit of some kind—say of impulse—should be called a *heave*.

We must now give some account of Heaviside's discoveries in the theory of electric signalling—that is, of telegraphs, telephones and cables. A theory of cable signalling had been given by W. Thomson as early as 1855, in which, taking account of the electrostatic capacity of the cable and of Ohm's law, he had obtained as the partial differential equation of electrical excitation in the cable

$$\frac{\partial^2 V}{\partial x^2} = RS \frac{\partial V}{\partial t}$$

where R and S are constants. This is the equation characteristic of processes of diffusion, being in fact the same as the well-known equation of the linear motion of heat in a solid conductor, and the various solutions which were discovered by Fourier are perfectly adapted for answering practical questions regarding the use of the cable. Thus if the end O of the wire is put for a very short time in communication with the battery, thereby receiving a quantity of electricity Q and is then insulated, the potential at a point x of the cable at time t is

$$V = \frac{Q R^{\frac{1}{2}}}{\pi^{\frac{1}{2}} S^{\frac{1}{2}} t^{\frac{1}{2}}} e^{-\frac{RSx^2}{4t}}.$$

There is no definite time of arrival for a signal, since a receiving instrument of perfect sensitiveness at any distance would (theoretically) record the signal as soon as it was sent out, while any practical receiving instrument would record its arrival only after an interval depending on the sensitiveness of the instrument. We can however show easily that, in the case of a signal such as the above the maximum effect is obtained at any given place x at a time $t = \frac{1}{2} RSx^2$. Thus the retardations of signals are proportional to the squares of the distances, and not to the distances themselves.

Unfortunately many of the electricians of the time misunderstood this statement: they seem to have pictured the current as knowing when it set

out on its journey how far it had to go and adjusting its rate of propagation accordingly, like a Trades Unionist practising ca' canny.

If on the other hand we suppose that a signal of a different type is sent out, the potential at the sending end being made to vary regularly according to a simple periodic law, then we find that the phases are propagated regularly with a continually-diminishing amplitude: but the rate of propagation of phase is different for waves of different frequencies, and the attenuation increases rapidly with the frequency, so that an initial disturbance consisting of waves of different frequencies superposed, such as human speech, would be hopelessly distorted in transmission.

Such was the state of the theory of cables, as it was presented in Maxwell's treatise in 1873. When as in the case of an Atlantic cable, it was only possible to get a small number of waves through per second, the influence of the self-induction of the cable was not great, and that is why Thomson's theory, which ignores self-induction altogether, was tolerably satisfactory. The first advance in the general theory was made when Heaviside in 1876 took account of self-induction, thus obtaining for the potential, in place of Thomson's equation, the partial differential equation

$$\frac{1}{LS}\,\frac{\partial^2 V}{\partial x^2}=\frac{R}{L}\,\frac{\partial V}{\partial t}+\frac{\partial^2 V}{\partial t^2}$$

which is now known as the *equation of telegraphy*, and might, I think, properly be called *Heaviside's equation*. When the term $\frac{R}{L}\frac{\partial V}{\partial t}$ is negligible, this equation becomes

$$\frac{1}{LS}\,\frac{\partial^2 V}{\partial x^2}=\frac{\partial^2 V}{\partial t^2}$$

which is the equation of propagation of waves such as those of sound or light, parallel to the axis of x with the constant velocity $1/\sqrt{LS}$ for disturbances of all periods, and with no attenuation, so that all signals are transmitted without any distortion whatever. This is evidently the ideal to aim at, and as Heaviside was the first to understand, it is to be sought by diminishing the co-efficient R/La; that is to say, by increasing the self-induction of the wire, we tend to make the retardations of phase equal for waves of different frequencies, and we also tend to make the alternations equal for waves of different frequencies, and therefore *inductance is a way of improving things*, of making the propagation of waves along wires similar to the propagation of waves in free aether.

Applying these ideas to the case when the waves correspond to the constituents of the human voice, Heaviside now saw clearly that the way to

attain telephonic speech through a long wire was to increase self-induction. Self-induction is, in fact, the same kind of benefit to the wave that inertia is to a body moving along against a viscous resistance.

Heaviside showed moreover that leakage of the current from the wire is not altogether undesirable: although it weakens the signals, it tends towards equalising the effects on waves of different frequencies, and so helps to prevent distortion.

To put the matter somewhat differently, we want to secure that the magnetic and electric energies shall be equal, as they must be for distortionless transmission; in the cables, the electric energy tends to excess: and the remedy is to increase the magnetic energy by increasing the self-induction, and also to diminish the electric energy by leakage.

Unfortunately, the officials of the British Post Office failed to see the merit of Heaviside's work. Mr. W. H. Preece, who was Electrician to the Post Office from 1877 to 1899, regarded self-induction as something harmful to telephony, and tried to avoid it as much as possible. Heaviside called attention publicly to the blunders of the Post Office, who for telephony had "put down conductors having a resistance of 45 ohms per mile, combined with large permittance and small inductance: and then to make the violation of electro-magnetic principles more complete, put the intermediate apparatus in sequence, so as to introduce as much additional impedence as possible." [1] Preece, however, seems to have been unmoved; and when Heaviside wrote a paper in conjunction with his brother Arthur, who was a subordinate of Preece in the Post Office Engineers, they were given to understand that "the official censor" ordered all Oliver's part to be left out.[2] Heaviside referred [3] to "a certain peculiar concurrence and concatenation of circumstances, rendering it impossible for me to communicate the practical applications of my theory, either *via* the *Society of Telegraphic Engineers and Electricians*, or four other channels, the resultant effect of which was to screen Mr. Preece from criticism." The editor of the *Electrician* "returned a short article on long distance telephony, which pointed out official errors in detail, and directed attention to the contrary results indicated by my theory, this paper having been in official hands. Perhaps it was thought that official views were so much more likely to be right that it was safe to decline the discussion of novel views in such striking opposition thereto. There seemed also to be an idea that official views, in virtue of their official nature, should not be controverted or criticised."

[1] *Electrician*, 7th October, 1887.

[2] *Electrical Papers* II, p. 324.

[3] *Electrician*, 19th October, 1888.

Preece was a high Government official, while Heaviside was a person without degrees or distinctions or position, living in a Camden-Town lodging-house. Later in life Preece prospered still more: he became a K. C. B., an officer of the Legion d'honneur, he had one house in Queen Anne's Gate and another at Wimbledon, while Heaviside was living in squalor in a country village. But if, as seems likely, Preece is known to posterity only by what is said about him in the works of Heaviside, time will have brought an ample revenge.

Heaviside's ideas on improving telephonic transmission by increasing the self-induction in the circuit were put into practice towards the end of the century in America by Pupin, who by inserting inductive coils at intervals along the line made telephony possible across the American continent. Later, continuous loading of the cable was introduced, precisely as Heaviside had suggested.

For the present, however, the outlook was dark. In November, 1887 he was "requested to discontinue" the series of papers in the *Electrician*. "A change of editor had occurred, and the new editor asked me to discontinue. He politely informed me that although he had made particular enquiries amongst students who would be likely to read my papers, to find if anyone did so, he had been unable to discover a single one."

However, the *Philosophical Magazine* was still open, and next year the *Electrician* relented, and published [1] an investigation of the electromagnetic effect of a moving charge. This marks a new stage in the development of Heaviside's fundamental notions, since now (convinced, doubtless, by the evidence that cathode-rays are torrents of negatively-charged particles) he abandoned the ultra-Maxwellian notion that "electricity" is merely a discontinuity in electric displacement and admitted it [2] as a quasi-substance capable of motion. A moving charge, he now asserts, "is equivalent to an electric current-element," the necessity for this may be seen on pure Maxwellian principles by simply considering that when a charge q is conveyed into any region, an equal displacement simultaneously leaves it through its boundary." So "the true current has three components, thus:

$$\text{curl}\ H=4\pi(C+\dot{D}+\rho u)$$

where H is the magnetic force, C the conduction-current, D the displacement, and ρ the volume-density of electrification moving with velocity u." He

[1] *Electrician*, 23rd Nov., 1888.

[2] As Maxwell himself had done, *Treatise* §§ 768-770 and, before him, Faraday, *Exper. Res.*, § 1644.

then proceeded [1] to calculate the field due to a charge q moving at speed u along the axis of z. Everything is symmetrical with respect to this axis; the lines of electric force are radial outwards from the charge, those of magnetic force are circles about the axis. Having thus settled the directions, it remains only to specify the intensities at any point P distant r from the charge, the line r making an angle θ with the axis. Denote by E the intensity of the electric, and by H of the magnetic force. Then if c is the permittivity and μ the inductivity, so that $\mu c v^2=1$ where v is the velocity of light, we have

$$\left\{\begin{aligned} cE &= \frac{\dfrac{q}{r^2}\left(1-\dfrac{u^2}{v^2}\right)}{\left(1-\dfrac{u^2}{v^2}\sin^2\theta\right)^{\frac{3}{2}}} \\ H &= cEu\sin\theta \end{aligned}\right.$$

In a paper within a few months afterwards,[2] he gave for the first time the all-important result that an electric point-charge e moving with velocity v in a magnetic field is acted on by a mechanical force equal to e multiplied by the vector-product of v and the magnetic induction. This may be regarded as the first appearance of the electron in electromagnetic theory.

The scientific papers which Heaviside had been publishing now for sixteen years had so far brought him little but neglect and ill-treatment from those in influential positions. But the tide of interest in the development of Maxwell's theory was slowly rising. The experimental work of Hughes in 1886 verified the theory of surface conduction along wires, which Heaviside had advanced a year previously, and this was followed in 1887-88 by the experimental work of Hertz and Lodge on electrical vibrations and electromagnetic waves, broadly confirming the truth of the theory regarding the propagation of disturbances along wires which Heaviside had worked out on the basis of Maxwell's doctrine. In January, 1889 Sir William Thomson, who was then President of the Institution of Electrical Engineers, delivered an address on "Ether, Electricity, and Ponderable Matter," [3] in which Heaviside's work was spoken of most highly, and at the same time Lodge wrote enquiring if he would allow his name to be put forward as a candidate for the Royal Society. Heaviside at first raised objections to this, but Lodge overcame them, and in June, 1891 he was elected a Fellow. Just about this

[1] *Electrician*, 7th December, 1888.
[2] *Phil. Mag.*, April, 1889, p. 324.
[3] Thomson's *Math. and Phys. Papers 3*, p. 484.

time [1] another change took place in his outward circumstances. The household at Camden Town, consisting of his father and mother and himself, resolved to leave London and settle at Paignton, on the coast of South Devon, where his brother Charles was engaged in the musical instrument business. Oliver never moved more than a few miles from Paignton during the rest of his life.

In 1891 the publishers of the *Electrician* proposed that a series of articles on "Electromagnetic Theory" just begun in that journal, should be brought out later in book-form. This raised the question of his earlier papers, the offprints of which were now exhausted. Eventually it was arranged to reprint the whole of his work up to the end of 1890, together with a Royal Society memoir of 1891 "on the forces, stresses, and fluxes of energy in the electromagnetic field" in two volumes, which appeared in 1892 under the title *Electrical Papers*. The sale was not very great. "They printed 750 copies," he told Bjerknes, "and had 359 copies left five years after. They disposed of these at reduced prices, 10*s*., 2*s* /6*d*., and then all the unbound remainder at 1*s*./6*d*., and so wiped it all out." "They can be picked up cheap," he wrote in 1901, "because the remainder was sold off in quires for a few pence per volume, on account of the deficiency in storage room. So look in the fourpenny boxes."

However little his own books might be read, he never ceased to study the writings of others. Forsyth's *Theory of Functions* and Whitehead's *Universal Algebra* were both published when he was in later middle life, and could scarcely be described as works on physics; but the copies which were found on his shelves after his death were full of his pencil annotations.

The year 1892 saw the beginning of a friendship with G. F. C. Searle of Cambridge, which was the chief support of his later days; and saw also the extension of a scientific correspondence which he carried on for many years with Fitzgerald, Lodge, Perry, Silvanus Thompson, Ayrton, and others. It is remarkable that (except for Searle) he had little to do with Cambridge: not that he was lacking in admiration for the University of Maxwell and Kelvin and Rayleigh and J. J. Thomson: "Good mathematicians," he said playfully, "when they die, go to Cambridge": but Cambridge was rather self-contained, and before 1890 his work had been scarcely known there; and when his merit was at last recognised, a most unfortunate dispute broke out between him and the Cambridge pure mathematicians, which alienated him from Cambridge for the rest of his life.

The subject of the dispute was what is now called the Operational Calculus. For many years Heaviside had been accustomed to use symbolic

[1] I have not been able to determine the precise date, but he was certainly living in St. Augustine's Road, Camden Town, in February, 1889, and at Paignton in June, 1892.

differential operators in his electromagnetic researches; having doubtless originally learnt them from the elementary works on Differential Equations, where, *e.g.*, the solution of such an equation as

$$\frac{d^2y}{dt^2}+k^2y=0$$

may be obtained by writing D for the operation of differentiating with respect to t, and then proceeding thus:

$$(\mathrm{D}^2+k^2)y=0, \text{ so } (1+k^2\mathrm{D}^{-2})y=\mathrm{D}^{-2}.\,0=a+bt;$$

so $$y=(1+k^2\mathrm{D}^{-2})^{-1}(a+bt)$$

$$=a.\ (1-k^2\mathrm{D}^{-2}+k^4\mathrm{D}^{-4}\ldots)\,.\,1+b.\ (1-k^2\mathrm{D}^{-2}+k^4\mathrm{D}^{-4}\ldots)t$$

$$=a\left(1-\frac{k^2t^2}{2!}+\frac{k^4t^4}{4!}-\ldots\right)+b\left(t-\frac{k^2t^3}{3!}+\ldots-\right)$$

$$=a\cos kt+\frac{b}{k}\sin kt.$$

Heaviside applied this method to establish a general "expansion theorem" as he called it [1] which may be stated thus:

Let a linear differential equation with constant coefficients be written

$$\mathrm{Z}\left(\frac{d}{dt}\right)i=\mathrm{E}$$

where t is the independent variable, i the dependent variable, E a given function of t, and where $\mathrm{Z}\left(\frac{d}{dt}\right)$ is an operator involving differentiations with respect to t. In the applications, i generally denotes the current at some point of an electrical system containing resistances, inductances, condensers, etc., and E denotes an impressed electromotive force applied at some other point of the system. Then *if i and E are zero previous to the instant $t=0$, and if at this instant E is suddenly increased to a finite value and maintained steadily at that value, the current i at time t is given by the equation*

$$i=\frac{\mathrm{E}}{\mathrm{Z}(0)}+\mathrm{E}\sum_{r=1}^{n}\frac{e^{p_rt}}{p_r\mathrm{Z}'(p_r)}$$

where $p_1, p_2, \ldots p_n$ are the roots of the equation $\mathrm{Z}(p)=0$. This formula, as he said, "goes straight to the final simplified result."

[1] *Phil. Mag.*, December, 1887, p. 479; *Electrician*, August 9, 1895.

These operational researches led him on naturally to *fractional differentiation*, *i.e.*, assigning a meaning to $\frac{d^n y}{dx^n}$ when n is not a positive integer. "There is a universe of mathematics," he said, "lying in between the complete differentiations and integrations." This is an old subject: Leibniz considered it in 1695 [1] and Euler in 1729 [2]: and indeed it was in order to generalise the equation

$$\frac{d^n(x^p)}{dx^n}=p(p-1)(p-2)\ldots(p-n+1)x^{p-n}$$

to fractional values of n that Euler invented the Gamma-Function, by the aid of which he arrived at the generalisation

$$\frac{d^n(x^p)}{dx^n}=\frac{\Gamma(p+1)}{\Gamma(p-n+1)}x^{p-n}$$

where n may be fractional. Since then there has been an almost continuous succession of papers about fractional derivatives.[3] Heaviside seems to have known nothing of them beyond a reference of a few lines in Thomson and Tait's *Natural Philosophy*:[4] but he carried the subject on original lines further, in some directions, than any of his predecessors. As an example of his method, consider the following problem:

A body with a plane face is initially at zero temperature; its face is thereafter maintained at constant temperature V_0; *to find the resulting temperature gradient at the face.*

Let V denote the temperature at distance x from the plane face at time t. Then the partial differential equation of conduction of heat in the body is

$$\frac{\partial^2 V}{\partial x^2}=h\frac{\partial V}{\partial t}$$

where h is constant. Write p for $\frac{\partial}{\partial t}$, so $\frac{\partial^2 V}{\partial x^2}=hpV$, and therefore

$$V=e^{-(hp)^{\frac{1}{2}}x}V_0$$

[1] Letter to the Marquis de l'Hôpital, September 30, 1695, and letter to John Bernoulli, December, 1695.

[2] Letter to Goldbach, October 13, 1729, printed in Fuss, *Corresp. Math.* I, p. 3.

[3] I have a list of 115 papers which I have noted.

[4] Vol. I, p. 197.

Thus the surface gradient$=\left(-\frac{\partial V}{\partial x}\right)_{x=0}=\left\{(hp)^{\frac{1}{2}}\,e^{-(hp)^{\frac{1}{2}}x}\,V_0\right\}_{x=0}$

$$=(hp)^{\frac{1}{2}}\,V_0=h^{\frac{1}{2}}\left(\frac{\partial}{\partial t}\right)^{\frac{1}{2}}V_0,$$

and since Euler's formula above gives

$$D^{\frac{1}{2}}\,.\,1=\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}=(\pi t)^{-\frac{1}{2}},$$

we have

$$\text{surface gradient}=\left(\frac{h}{\pi t}\right)^{\frac{1}{2}}V_0.$$

As a second example of Heaviside's methods, let us take another problem relating to the conduction of heat in an infinite solid with a plane face : let us consider the solution of the equation

$$\frac{\partial^2 V}{\partial x^2}=k\frac{\partial V}{\partial t} \qquad \dots \quad (1)$$

subject to the initial condition

$$V=0 \text{ when } t=0$$

and to the boundary condition

$$V_0-V=h\frac{\partial V}{\partial x}\text{ when } x=0,$$

where V_0 and h are constants. Let it be required to find the value of V when $x=0$, and let us denote this function by V_1.

We extract a kind of operational square root of equation (1), converting it into

$$\frac{\partial V}{\partial x}=\left(k\frac{\partial}{\partial t}\right)^{\frac{1}{2}}V,$$

whereby we are enabled to transform the boundary condition into the form

$$V_0-V_1=h\left(k\frac{\partial}{\partial t}\right)^{\frac{1}{2}}V_1.$$

So

$$V_1 = \frac{1}{1+\left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}}} V_0 \text{ where } a = k\,h^2, \qquad \ldots \quad (2)$$

whence

$$V_1 = \left\{1 - \left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}} + a\frac{\partial}{\partial t} - \left(a\frac{\partial}{\partial t}\right)^{\frac{3}{2}} + \left(a\frac{\partial}{\partial t}\right)^2 - \ldots\right\} V_0$$

$$= V_0\left[1 - \left(\frac{a}{\pi t}\right)^{\frac{1}{2}}\left\{1 - \frac{a}{2t} + 1.3.\left(\frac{a}{2t}\right)^2 - \ldots\right\}\right] \quad \ldots \quad (3)$$

This asymptotic expansion is useful when t is large. When t is small on the other hand, we have from (2)

$$V_1 = \frac{1}{\left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}}}\left\{1 + \frac{1}{\left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}}}\right\}^{-1} V_0$$

which on expanding gives

$$V_1 = ZV_0\left(\frac{t}{a\pi}\right)^{\frac{1}{2}}\left\{1 + \frac{2t}{3a} + \frac{1}{3\,5}\left(\frac{2t}{a}\right)^2 + \frac{1}{3.5.7.}\left(\frac{2t}{a}\right)^3 + \ldots\right\}$$

$$+ V_0\left(1 - e^{\frac{t}{a}}\right) \qquad \ldots \quad (4)$$

The solutions (3) and (4) are as a matter of fact perfectly correct.[1]

One can imagine the feelings of the professional pure mathematicians as they read this kind of thing At that time the most influential of them were trying to raise the standard of "rigour"—to move away from the happy old easy-going Todhunter period to the style which the Germans had adopted under the influence of Weierstrass: and the "operation" of extracting the square root of the process of partial differentiation seemed worse than anything in Todhunter—a kind of mathematical blasphemy. Not long afterwards

[1] As is well-known, the expansions (3) and (4) both represent the function

$$V_0\left\{1 - \frac{2}{\pi^{\frac{1}{2}}} e^{\frac{t}{a}} \int_{\left(\frac{t}{a}\right)^{\frac{1}{2}}}^{\infty} e^{-s^2}\,ds\right\},$$

(3) being the asymptotic expansion and (4) the convergent series.

one of them told me what happened. "There was a sort of tradition," he said, "that a Fellow of the Royal Society could print almost anything he liked in the 'Proceedings' without being troubled by referees : but when Heaviside had published two papers on his symbolic methods, we felt that the line had to be drawn somewhere, so we put a stop to it." Refused publication in the *Proc. R. S.*, he continued in the *Electrician*; but the wound was sore, and the alienation from the Cambridge mathematicians lasted all this life.[1] He did not fail, however, to defend his views in public. "Mathematics is an experimental science," he said, "and definitions do not come first, but later on." "The Euclidean logical way of development is out of the question. That would mean to stand still. First get on, in any way possible, and let the logic be left for later work." "It is not so long ago since mathematicians of the highest repute could not see the validity of investigations based upon the use of the algebraic imaginary. The results reached were, according to them, to be regarded as suggestive merely, and required proof by methods not involving the imaginary." "The use of operators frequently effects great simplifications, and the avoidance of complicated evaluations of definite integrals. But then the rigorous logic of the matter is not plain! Well, what of that? Shall I refuse my dinner because I do not fully understand the process of digestion?" "There is an idea widely prevalent that in mathematics, unless you follow regular paths, you do not prove anything: and that you are bound to understand fully and prove rigorously everything as you go along. This is a most pernicious doctrine, when applied to imperfectly explored regions. Does anybody fully understand anything?" "Three of the pernicious results of overmuch rigour may be mentioned here. First, its enfeebling action on the mind. Secondly, it leads to the omission from mathematical works of the most interesting parts of the subject, because the authors cannot furnish rigorous proofs. Thirdly, it leads to an inability to recognise the good that may be in other men's work, should it be unconventional, and be devoid of rigorous pretence."

Looking back on the controversy after thirty years, we should now place the Operational Calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the Tensor Calculus as the three most important mathematical advances of the last quarter of the nineteenth century. Applications, extensions and justifications of it constitute a considerable part of the mathematical activity of to-day.[2]

[1] "Even men who are not Cambridge mathematicians deserve justice, which I very much fear they do not always get" (*Electrician*, Dec. 14,1892).

[2] I am thinking of the recent work of H. Jeffreys, Norbert Wiener, Bromwich, Parson, Murnaghan, March and others.

Besides his troubles with the "scienticulists" and "mathematicians of the Cambridge or conservatory kind, who look the gift-horse in the mouth and shake their heads," as he described them, Heaviside had the constant trouble of poverty. In February, 1894 three of his friends[1] wrote him a joint letter, asking his consent to a proposal to obtain for him a grant from the Royal Society's Scientific Relief Fund. But he looked on this as "charity," and refused to allow the matter to go forward. However, in the spring of 1896 the same friends, with the support of Lords Rayleigh and Kelvin, induced Mr. Balfour to recommend him for a Civil List Pension of £120 a year, and this he accepted.[2]

During these years he had been living at Paignton with his parents: but his mother died in 1894 and his father in 1896, and he was left alone. Early in 1897 he removed to "Bradley View," Newton Abbot—a small town, seven miles from Paignton, where he lived until 1908, at first with a housekeeper, and then by himself. He never attended scientific gatherings, but sometimes another mathematical physicist would journey to Devonshire to see him: a visit of this kind from Fitzgerald in September, 1898 gave him great pleasure. For exercise he cycled; and in July, 1900 had a slight accident caused by running over a hen. His papers in the *Electrician* from 1891 to 1898 were now collected under the title *Electromagnetic Theory* in two volumes, of which the first was published in 1893 and the second in 1899; and further honours came to him: he was elected an Honorary Member of the Manchester Literary and Philosophical Society in 1894, and a Foreign Honorary Member of the American Academy of Arts and Sciences in 1899.

In the latter year an enquiry of Fitzgerald's turned his attention to a new problem. "Have you," wrote Fitzgerald, "worked out the propagation of waves round a sphere?—a case of this is troubling speculators as to the possibility of telegraphing by electromagnetic free waves to America. It is evidently a question of diffraction, and I think must be soluble."

Heaviside's work on the guidance of electromagnetic waves by conducting wires suggested a physical solution. "A wire," he said in 1902, "seems to guide a disturbance round a corner, by holding on to the tubes of displacement by their ends......There is something similar in 'wireless' telegraphy. Sea water has quite enough conductivity to make it behave as a conductor for Hertzian waves, and the same is true in a more imperfect manner of the earth. There is another consideration. There may possibly be a sufficiently conducting layer in the upper air. If so, the waves will, so to speak, catch

[1] Fitzgerald, Lodge and Perry.

[2] About twelve years afterwards it was increased.

on to it more or less. Then the guidance will be by the sea on one side and the upper layer on the other." The permanently ionised layer in the upper air here introduced was afterwards called by Eccles the "Heaviside layer," and under that name is familiar to all students of "wireless."[1]

Geophysics and Meteorology were indeed among the chief interests of the latter part of his life. "I am very familiar," he wrote in 1921, "with the surface of discontinuity separating streams of air, which is, it seems to me, not a surface really, but may have considerable depth, and have made some excellent predictions here. Said I to a bobby who came one evening with a summons, 'Look at that sky!' 'Oh! beautiful,' he answered 'Do you know what it means? There's going to be *very rough weather*. It may not come till to-morrow, or it may come in the night.' Bobby did not seem to care. It came in the night, and did hundreds of pounds' worth of damage on the little bathing beach near the station."

Much of his work after 1900 was concerned with the radiation from electrons describing circular and other orbits, the general theory of an electric charge in variable motion, the generation of pulses, and the theories of other electrical writers, *e.g.*, Helmholtz, Lorentz, and Larmor. In reviewing these he expressed a strong dislike to the Principle of Least Action, "which," he said, "in spite of its name, has the remarkable property of increasing the amount of work to be done." In 1902 he wrote the article "Telegraphy, Theory," for the "New Volumes" of the *Encyclopaedia Britannica*; and in 1912 nearly all the papers which he had published since 1899 were collected in a third and final volume of *Electromagnetic Theory* The three volumes were reprinted in 1922, showing that at last—at seventy-two—he had a reading public: and they were reprinted again in 1925.

In the latter part of his life more honours were offered The Council of the Royal Society resolved to award him the Hughes Medal in 1904: but he declined it privately and the announcement was not made public, another recipient being chosen. In 1905 the University of Göttingen, where Weber and Gauss had worked, conferred on him the degree of Ph.D. *honoris causa*. The Institution of Electrical Engineers made him an Honorary Member in 1908, and the American Institute of Electrical Engineers in 1918. He was the first recipient of the Faraday Medal of the Institution of Electrical Engineers in 1923, the President of the Institution personally journeying to Devonshire to present it.

[1] Sir J. Larmor has given reasons for believing that it is the free oscillation of the ions in the layer, undisturbed by collisions, which bends the waves.

The years 1919-1920 saw an extraordinary display of popular interest in the theory of General Relativity, which had been discovered[1] by Einstein and Hilbert in 1915, and vindicated triumphantly by the eclipse observations of May, 1919. Heaviside disliked it. "The Einstein enthusiasts," he wrote in March, 1920, "are very patronizing about the "classical" electromagnetics and its ether, which they have abolished. But they will come back to it by and by. Though it leaves gravity out in the cold, as I remarked about 1901 (I think), gravity may be brought in by changes in the circuital laws, of practically no significance save in some very minute effects of doubtful interpretation (so far). But you must work fairly with the Ether, and Forces, and Momentum, etc. *They* are the realities, without Einstein's distorted nothingness."

In 1908 he had removed from Newton Abbot to "Homefield," Lower Warberry Road, Torquay, where the rest of his life was spent: at first a relative, a Miss Way, kept house for him, but for the last eight years the old man lived quite alone, deaf, eccentric, a butt for the boys of the neighbourhood, and often in arrears with his rates. "As regards myself," he wrote to Bjerknes at the beginning of 1921, "I was very ill indeed after writing to you, and had great financial trouble as well, and had to borrow many hundreds of pounds to pay the bank and others, to avoid being sold up, and left homeless. I got long credit from the Gas Co., but they tired of that, and began their "cutting off the gas" threats again. A legacy of £155 came in just in time to help, and I paid up on September 30th. Now they are at me again." From August, 1921 to October, 1922 the gas was cut off entirely from his house. A distinguished physicist who went to see him in the Christmas Vacation of 1921-22, on arriving at Torquay, wrote that he would call next day: but on calling found the door closed and a note outside to say that Heaviside was not disposed to see visitors, having gone to bed to keep himself warm. The front door was covered with papers which he had stuck up—an invitation from the President of the Royal Society to a Conversazione, an advertisement of "Twink," notices from the Gas Company about debts and cutting off gas, a judgment against him in the County Court, Poor Rate notices, etc. "I have had a dreadful bad winter," he wrote in June, 1922, "in bed nearly all the time, no gas, cold house, cold food, lumbago of course, but the worst was rheumatic gout. Could not wear boots at all. Could not get proper bed socks to walk about in. Buried under all the blankets I have. Now and then I scribbled a sort of diary about my persecution by the Poor and the Gas and others." Even in these painful circumstances, however, his scientific correspondence was enlivened by references to passing events

[1] That is to say, the field-equations had been given

with frequent quotations or impromptu verses. The United States claim for repayment of the English debt in 1923 led to some scathing remarks on American greed and greed in general, ending with

> "Eat slowly. Only men in rags, or gluttons deep in sin
> Mistake themselves for carpet bags, and shove the vittles in."

* * *

At the last his friends took him into a nursing home, where he died on February 3rd, 1925, at the age of 75. He lies buried in the grave of his father and mother in Paignton Cemetery.

ÜBER DIE HAUPTSCHNITTE EINES POLYDIMENSIONALEN WÜRFELS

VON

A. SOMMERFELD

[*Read April 7, 1929*]

In einer Sitzung der Mathematischen Gesellschaft zu Calcutta, an der ich teilzunehmen das Vergnügen hatte, erwähnte der Vorsitzender, Professor Ganesh Prasad, eine Ableitung des Gaussischen Fehlergesetzes, die ich in der Boltzmann-Festschrift [1] veröffentlich habe. Diese Ableitung zeigt in besonders anschaulicher Weise, wie sich bei n voneinander unabhängigen Elementarfehlern als Wahrscheinlichkeit für den aus ihnen additiv resultierenden Gesamt-Fehler eine Curve ergibt, die der Gaussischen Curve mit wachsendem n immer ähnlicher wird und für $n \longrightarrow \infty$ in sie übergeht. Die folgenden Zeilen ergänzen die frühere Darstellung und berichtigen einen dort vorkommenden Zahlenfactor.

1. *Formulierung des geometrischen Problems.* Es werde wie früher angenommen, dass alle Elementarfehler, x_1, x_2, x_3,...x_n, dasselbe einfachste Gesetz befolgen, dass sie nämlich zwischen den Grenzen $-\frac{1}{2}$ und $\frac{1}{2}$ die Wahrscheinlichkeits-Dichte 1 besitzen und diese Grenzen niemals überschreiten können. Der resultierende Fehler ist

$$x = x_1 + x_2 + x_3 + \ldots x_n. \tag{1}$$

Gefragt wird nach der Wahrscheinlichkeit $y\,dx$, dass der resultierende Fehler zwischen x und $x+dx$ liege.

Wir konstruieren im n-dimensionalen Raum der $x_1 \ldots x_n$ einen Würfel von der Kantenlänge 1; als Haupt-Diagonale bezeichnen wir die Verbindungslinie der Würfel-Eckpunkte $\frac{1}{2}$, $\frac{1}{2}$,...und $-\frac{1}{2}$, $-\frac{1}{2}$... Gl.(1) stellt dann eine zu dieser Hauptdiagonalen normale "Ebene," *d.h.* einen ebenen R_{n-1} dar, sofern wir x festhalten. Diese "Ebene" schneidet aus dem Würfel ein gewisses $(n-1)$-dimensionales Gebiet aus, das wir in der Überschrift "Hauptschnitt" genannt haben. Seine Grösse sei, als Funktion der durch x gegebenen Lage der Hauptebene betrachtet, $f(x)$. Die n-dimensionale

[1] Leipzig, 1904, bei Joh. Ambr. Barth, vgl. pag. 848.

Schicht dV, die von den Hauptschnitten x und $x+dx$ zusammen mit Teilen der Würfelflächen begrenzt wird, ist dann

$$dV = \frac{1}{\sqrt{n}} f(x) dx. \tag{2}$$

Man überlegt nämlich leicht, dass der Abstand der beiden Hauptschnitte als Längenelement auf der Haupt-Diagonalen gegeben ist durch $dx/\sqrt{n}$. Die Wahrschein lichkeits-Dichte y wird dann

$$y = \frac{dV}{dx} = \frac{1}{\sqrt{n}} f(x). \tag{3}$$

Es kommt uns darauf an, $f(x)$ zu berechnen. Dies geschah früher schrittweise auf elementar-geometrischen Wege und soll jetzt allgemein analytisch durchgeführt werden.

2. *Integral Darstellung von* $f(x)$. Das Volumen des n-dimensionalen Würfels ist

$$V = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_1 \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_2 \dots \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_n.$$

Wir erhalten daraus die genannte Schicht dV, wenn wir einen diskontinuierlichen Dirichlet 'schen, Faktor D hinzufügen, der gleich 1 ist, wenn

$$x < x_1 + x_2 + x_3 + \dots x_n < x + dx$$

und für alle anderen Werte von x_1, x_2,...verschwindet. Er kann nach Fourier dargestellt werden durch [1]

$$D = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{x}^{x+dx} e^{i\lambda(x_1 + x_2 + \dots x_n - \xi)}\, d\xi. \tag{4}$$

Also

$$dV = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_1 \dots \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_n D \tag{5}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_1 \dots \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_n e^{i\lambda(x_1 + x_2 + \dots + x_n - x)}\, dx$$

[1] Die Benutztung der Exponential-Funktion an Stelle des sonst ublichen Cosinus vereinfacht die folgende Rechnung.

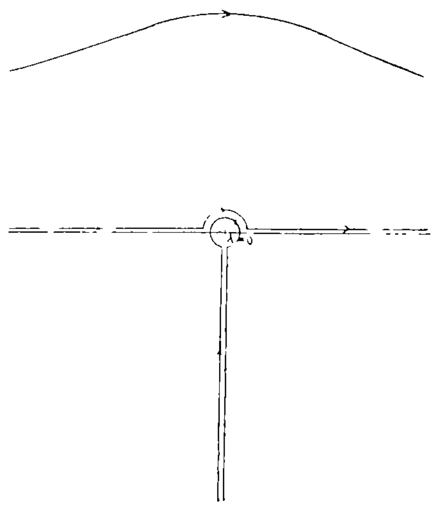


Fig 1

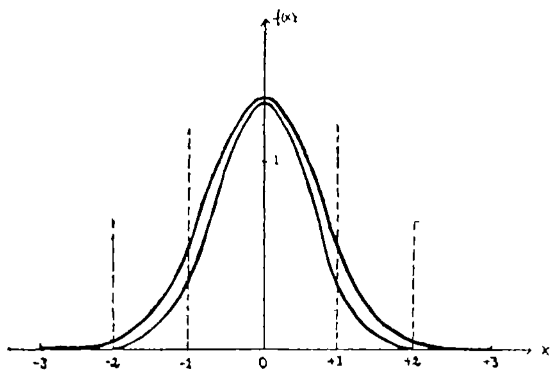


Fig. 2

In der letzten Formel haben wir bereits die Reihenfolge der Integrationen nach λ und $x_1, \ldots x_n$ vertauscht und die Integration nach ξ für hinreichend kleines $d\tau$ näherungsweise ausgeführt, was nach Vertauschung der Integrationsfolge erlaubt ist. Aus (5) und (3) folgt nunmehr

$$(6) \qquad f(x) = \frac{\sqrt{n}}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_1 \ldots \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_n e^{i\lambda(x_1 + \ldots + x_n - x)}.$$

Wir führen zunächst die Integration nach x_n aus:

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_n e^{i\lambda(x_1 + \ldots + x_n - x)} = e^{i\lambda(x_1 + \ldots + x_{n-1} - x)} \frac{e^{i\lambda/2} - e^{-i\lambda/2}}{i\lambda}$$

Integrieren wir diesen Ausdruck weiter nach x_{n-1}, zwischen $-\frac{1}{2}$ und $+\frac{1}{2}$, so entsteht ersichtlich

$$e^{i\lambda(x_1 + \ldots + x_{n-2} - x)} \left\{ \frac{e^{i\lambda/2} - e^{-i\lambda/2}}{i\lambda} \right\}^2$$

Mithin folgt aus (6) durch fortgesetzte Integration nach $x_{n-2} \ldots x_2, x_1$:

$$(7) \qquad f(x) = \frac{\sqrt{n}}{2\pi} \int_{-\infty}^{+\infty} d\lambda \; e^{i\lambda x} \left\{ \frac{e^{i\lambda/2} - e^{-i\lambda/2}}{i\lambda} \right\}^n.$$

Hier würde der Exponent $i\lambda x$ zunächst mit negativem Vorzeichen auftreten; die Vertauschung von λ mit $-\lambda$ gestattet aber, wie geschehen, den Exponenten mit dem positiven Vorzeichen zu schreiben, Daraus folgt zugleich, dass $f(x)$ in x gerade is, wie übrigens nach der Würfel-Symmetrie sich von selbst versteht.

Eine mit (7) äquivalente Formel wurde früher dazu benutzt, um den Grenzübergang $n \longrightarrow \infty$ auszuführen. Wir interessieren uns jetzt für die Darstellung von $f(x)$ bei beliebigem endlichen n.

3. *Algebraische Darstellung von* $f(x)$. Die unter dem Integralzeichen in (7) stehende Funktion ist für alle endlichen λ stetig und analytisch. Entwickeln wir aber die $n$te Potenz binomisch, so tritt in jedem einzelnen Glied der Entwickelung an der Stelle $\lambda=0$ ein Pol n ter Ordnung auf. Wir müssen dann diese Stelle mit unserem (ursprünglich reell gedachten) Integrationswege umgehen, indem wir um sie einen Halbkreis z.B. in der

positiv-imaginären Halbebene, vgl. Fig. 1, beschreiben. In allen Gliedern, die beim Ausmultiplizieren der Exponentialfunktionen in Exponenten von e mit einem *positiven* Faktor von $i\lambda$ behaftet sind, können wir daraufhin den Integrationsweg nach dem Unendlichen der positiven Halbebene hinüberziehen, wobei diese Glieder verschwinden. Ist aber der genannte Faktor in Exponenten *negativ*, so müssen wir den Integrationsweg nach der negativen Halbebene verlegen, wobei er an dem Pole $\lambda=0$ hängen bleibt und einen von Null verschiedenen Wert liefert. Wir suchen daher solche Glieder in der Binomial-Entwickelung auf, für die der Faktor von $i\lambda$ *negativ* wird.

(*a*) Den algebraisch kleinsten Wert hat dieser Faktor in dem letzten Gliede der Binomialreihe, welches die Exponential-Funktion liefert:

$$e^{i\lambda(x-n/2)} \tag{8}$$

Ist daher x grösser als $n/2$, so giebt es kein Glied mit negativem Exponenten und wir haben

$$f(x)=0, \quad x>n/2. \tag{9}$$

(*b*) Ist x kleiner als $n/2$ aber grösser als $n/2-1$, so liefert allein das Glied (8) einen Beitrag, welcher sich nach der Residuen-Methode als Faktor von $1/\lambda$ bei der Potenz-Entwickelung um den Pol $\lambda=0$ folgendermassen ergiebt:

$$(-1)^n\int\frac{e^{i\lambda(x-n/2)}}{(i\lambda)^n}\,d\lambda=(-1)^n\int\frac{\ldots+\frac{1}{(n-1)!}(i\lambda)^{n-1}(x-n/2)^{n-1}+\ldots}{(i\lambda)^n}\,d\lambda$$

$$=\frac{-1}{(n-1)!}\left(\frac{n}{2}-x\right)^{n-1}\int\frac{d\lambda}{i\lambda}=\frac{2\pi}{(n-1)!}\left(\frac{n}{2}-x\right)^{n-1},$$

indem nämlich das Integral im vorletzten Gliede der Gleichung -2π wird, vgl. den Umlaufssinn in Fig. 1. Nach (7) ergiebt sich also:

$$f(x)=\frac{\sqrt{n}}{(n-1)}\left(\frac{n}{2}-x\right)^{n-1}, \quad \frac{n}{2}-1<x<\frac{n}{2}. \tag{10}$$

(*c*) Nimmt x weiter *ab*, so giebt auch das vorletzte Glied der Binomialreihe zu einem negativen Exponenten Anlass, dagegen das nächst folgende Glied noch nicht, sofern x grösser als $n/2-2$ bleibt. Dieses vorletzte Glied lautet

$$(-1)^{n-1}\binom{n}{1}\int\frac{e^{i\lambda\left(x-\frac{n-2}{2}\right)}}{(i\lambda)^n}\,d\lambda$$

und liefert bei entsprechender Behandlung:

$$-\binom{n}{1}\frac{2\pi}{(n-1)!}\left(\frac{n-2}{2}-x\right)^{n-1}.$$

Indem man das schon berechnete letzte Glied der Binomial-Reihe hinzufügt, erhält man jetzt:

$$(11)\quad f(x)=\frac{\sqrt{n}}{(n-1)!}\left\{\left(\frac{n}{2}-x\right)^{n-1}-\binom{n}{1}\left(\frac{n}{2}-x-1\right)^{n-1}\right\},$$

$$\frac{n}{2}-2<x<\frac{n}{2}-1.$$

(*d*) Wir können daraufhin den allgemeinen Ausdruck für das Intervall

$$(12)\qquad \frac{n}{2}-k-1<x<\frac{n}{2}-k$$

(k ganzzahlig) hinschreiben. Zur Abkürzung führen wir statt x die Grösse ein:

$$(13)\qquad u=\frac{n}{2}-x,$$

die also nicht wie x von Mittelpunkte des Würfels, sondern vom Endpunkte $\frac{1}{2}, \frac{1}{2}, \ldots$ der Diagonalen gezählt wird. Dann ergiebt sich für das Intervall (12):

$$(14)\quad f(x)=\frac{\sqrt{n}}{(n-1)!}\left\{u^{n-1}-\binom{n}{1}(u-1)^{n-1}+\binom{n}{2}(u-2)^{n-1}-\ldots\right.$$

$$\left.\ldots+(-1)^{k}\binom{n}{k}(u-k)^{n-1}\right\}.$$

Der Zahlenfaktor N in den Gln. (2) und (3) der ursprünglichen Publikation (Boltzmann-Festschrift l.c.) ist daraufhin zu korrigiren.

4. *Einige Folgerungen.* Gl. (14) lehrt zunächst, dass die verschiedenen Curvenstücke, aus denen sich $f(x)$ zusammensetzt, an den Grenzpunkten

$$x=\frac{n}{2}-k,\ d.h.\ u=k$$

stetig aneinander anschliessen und dass auch die $n-2$ ersten Differentialquotienten dort stetig verlaufen. In der Tat unterscheidet sich Gl (14) von der im vorangehenden Intervall

$$\frac{n}{2}-k<x<\frac{n}{2}-k+1$$

geltenden Formel nur durch das letzte Glied von (14), und dieses verschwindet für $u=k$ zusammen mit seinen $n-2$ ersten Ableitungen. Hierin kommt die *fortschreitende Regularisierung* der Funktion $f(x)$ bei wachsendem n zum Ausdruck, die schliesslich für $n=\infty$ zu der durchweg analytischen Funktion

des Gaussischen Gesetzes führt. Z.B. wird für $n=1$ die Funktion $f(x)$ an den Grenzpunkten $x=\pm\frac{1}{2}$ selbst unstetig, für $n=2$ nur die Tangente von $f(x)$ (Grenzpunkte $x=+1, 0, -1$), für $n=3$ erst die Krümmung (Grenzpunkte $x=\pm\frac{3}{2}, \pm\frac{1}{2}$) *u.s.f.* (Man vergleiche die früher gezeichneten Figuren l.c.) Die abwechselnden Vorzeichen in (14) entsprechen offenbar den *Abstumpfungen* welche die Schnittfigur erfährt, jedesmal, wenn wir mit unserem Hauptschnitt eine Würfelecke überschreiten.

Ihr Maximum erreicht die Funktion $f(x)$ für $x=0$, d.h. nach (13) für $u=n/2$. Der Wert dieses Maximums, also die *scheinbare Grösse* des aus der Diagonalen-Richtung gesehenen n-dimensionalen Würfels ist bei geradem $n=2m$:

$$f(o)=\frac{\sqrt{n}}{(n-1)!}\left\{\left(\frac{n}{2}\right)^{n-1}-\binom{n}{1}\left(\frac{n}{2}-1\right)^{n-1}+\ldots+(-1)^{m-1}\binom{n}{m-1}\right\}$$

während bei ungeradem $n=2m+1$ das letzte Glied lautet:

$$(-1)^m\binom{n}{m}\left(\tfrac{1}{2}\right)^{n-1}.$$

Gl. (14) gilt nicht nur bei positiven, sondern auch bei negativen x, in welchem Falle $k>n/2$ ist, sofern nur $x>-n/2$ bleibt. Für $x<-n/2$ wird natürlich, ebenso wie für $x>+n/2$, $f(x)=0$. Man sieht dies aus unserer Integral-Darstellung (7) am einfachsten, wenn man, was erlaubt ist, mit dem Integrationswege nach der negativen-imaginären λ-Halbebene ausweicht.

Der gerade Charakter von $f(x)$ hat nach (14) die merkwürdige Formel zur Folge:

$$u^{n-1}-\binom{n}{1}(u-1)^{n-1}+\ldots(-1)^k\binom{n}{k}(u-k)^{n-1}=$$

$$v^{n-1}-\binom{n}{1}(v-1)^{n-1}+\ldots(-1)^l\binom{n}{l}(v-l)^{n-1},$$

$$u=\frac{n}{2}-x,\ v=\frac{n}{2}+x,\ l=n-k.$$

Fig. 2 stellt $f(x)$ für $n=4$ und $n=6$ dar. Im Falle $n=4$ besteht die Figur aus 4 Parabeln 3 ter Ordnung, die sich in den Grenzpunkten $x=0$, $x=\pm 1$ an einander und in den Grenzpunkten $x=\pm 2$ an die Abscissenaxe stetig und mit stetiger Tangente und Krümmung anschliessen. Das Maximum für $x=0$ beträgt $4/3$, die Ordinaten für $x=\pm 1$ sind gleich $1/3$. Die Aehnlichkeit mit den Gaussischen Fehlergesetz ist bereits bei diesem kleinen Werte von n augenfällig. Noch grösser ist sie im Falle $n=6$, wo sich die Curve von $x=-3$ bis $x=+3$ erstreckt und für $x=0$ das Maximum $\frac{11\sqrt{6}}{20}$ erreicht.

UEBER UNENDLICHE REIHEN UND ABSOLUT-ADDITIVE MENGENFUNKTIONEN

VON

HANS HAHN (IN WIEN)

[*Read August 19, 1928*]

Wir betrachten im folgenden unendliche Reihen aus positiven gegen 0 konvergierenden Gliedern a_ν :

$$\lim_{\nu \to \infty} a_\nu = 0; \tag{1}$$

wir setzen :

$$\sum_{\nu \geq 1} a_\nu = g; \tag{2}$$

die Konvergenz der Reihe (2) wird nicht vorausgesetzt ; ist diese Reihe divergent, so ist $g = +\infty$ zu setzen. Ist $\nu_1, \nu_2, \nu_3, \ldots \nu_i, \ldots$ eine Folge wachsender natuerlicher Zahlen, so nennen wir die Reihe $\sum_{i \geq 1} a_{\nu_i}$ eine *Teilreihe* der Reihe (2). Wir behandeln nun die Frage, unter welchen Umstaenden *jede* der Ungleichung $0 < x \leq g$ genuegende Zahl x durch eine Teilreihe von (2) dargestellt werden kann, und unter welchen Umstaenden jedes solche x nur durch eine einzige Teilreihe von (2) dargestellt wird.

Die Beantwortung dieser Fragen ist keinesfalls schwierig; wenn ich sie hier trotzdem mitteile, so geschieht es einerseits, weil ich diese einfachen Ueberlegungen in den ueblichen Lehrbuechern nicht gefunden habe, andrerseits, weil mir von hieraus der natuerlichste Weg zu einem tiefer liegenden Satze aus der Theorie der absolut-additiven Mengenfunktionen zu fuehren scheint, naemlich zu dem von W. Sierpinski (*Fund. math.* 3, 1922, p. 240) und Fréchet (*Fund. math.* 4, 1923, p. 364) bewiesenen "Zwischenwertsatze," dass unter einer gewissen einfachen Voraussetzung eine absolut-additive Mengenfunktion ϕ, die fuer zwei Teile A und B einer Menge M die Werte ϕ (A) und ϕ(B) annimmt, auch jeden zwischen ϕ (A) und ϕ (B) gelegenen Wert fuer mindestens einen Teil von M annimmt.

§ 1

Wir betrachten die Reihe (2); ihre Glieder a_ν seien positive Zahlen, die der Bedingung (1) genuegen. In ueblicher Weise bezeichnen wir mit s_n und r_n $(n\geq 1)$ die n-te Teilsumme und den Rest nach dem n-ten Gliede von (2):

$$(3) \qquad s_n = \sum_{1\leq\nu\leq n} a_\nu\,; \qquad r_n = \sum_{\nu>n} a_\nu \ (n\geq 1)\,;$$

fuer $n=0$ ergaenzen wir diese Definition durch:

$$(4) \qquad s_0 = 0.$$

Ebenso bezeichnen wir mit $s_{k,n}$, $r_{k,n}$ die Teilsummen und Reste von r_k, also:

$$(5) \qquad s_{k,n} = \sum_{k<\nu\leq k+n} = r_{k,n}\,, \qquad \sum_{\nu>k+n} a_\nu = r_{k+n} \ (n\geq 1)$$

$$(6) \qquad s_{k,0} = 0.$$

Wir beweisen nun den Satz:

I. *Sei (2) eine Reihe aus positiven, der Bedingung (1) genuegenden Gliedern und sei* $r_n\geq a_n$ für $n\geq 1$; *dann gibt es zu jeder, der Ungleichung* $0<x\leq g$ *genuegenden Zahl* x *eine Teilreihe von (2), fuer die* $\sum_{i\geq 1} a_{\nu_i} = x$ *ist.*

Dies trifft sicher zu fuer $x=g$, da dann die Reihe (2) selbst die gewuenschte Teilreihe ist. Sei also $0<x<g$. Nach (2) und (3) gilt: $s_n \longrightarrow g$; somit gibt es in der Folge $s_1, s_2 \ldots$ ein erstes Glied s_{n_0}, das $\geq x$ ist. Dann ist:

$$(7) \qquad s_{n_0-1} < x \leq s_{n_0} \ (= s_{n_0-1} + a_{n_0}).$$

Da nach Annahme $r_{n_0}\geq a_{n_0}$, ist auch

$$s_{n_0-1} + r_{n_0} \geq x.$$

Gilt hierin das $=$-Zeichen, so haben wir in $s_{n_0-1} + r_{n_0}$ eine Teilreihe von (2) vor uns, die $= x$ ist, und die Behauptung ist bewiesen. Sei also $s_{n_0-1} + r_{n_0} > x$, somit nach (7):

$$s_{n_0-1} < x < s_{n_0-1} + r_{n_0}.$$

Nach (3), (5) gilt $\lim\limits_{n\to\infty} s_{n_0,n}=r_{n_0}$, es gibt also in der Folge $s_{n_0,1}$, $s_{n_0,2}$, $\ldots s_{n_0,n}$ ein erstes Glied $s_{n_0,\,n_1}$, fuer das $s_{n_0-1}+s_{n_0,\,n_1}\geq x$ ist. Dann ist:

$$(8)\qquad s_{n_0-1}+s_{n_0,n_1-1}<x\leq s_{n_0-1}+s_{n_0,\,n_1}\ (=s_{n_0-1}$$
$$+s_{n_0,\,n_1-1}+a_{n_0+n_1}).$$

Da nach Annahme $r_{n_0+n_1}\geq a_{n_0+n_1}$, ist auch: $s_{n_0-1}+s_{n_0,n_1-1}+r_{n_0+n_1}\geq x$. Gilt hierin das $=$-Zeichen, so haben wir in $s_{n_0-1}+s_{n_0,n_1-1}+r_{n_0+n_1}$ ein Teilreihe von (2) vor uns, die $=x$ ist, und die Behauptung ist bewiesen. Sei also $s_{n_0-1}+s_{n_0,\,n_1-1}+r_{n_0+n_1}>x$, also nach (8)

$$s_{n_0-1}+s_{n_0,\,n_1-1}<x<s_{n_0-1}+s_{n_0,\,n_1-1}+r_{n_0+n_1}.$$

Wegen $\lim\limits_{n\to\infty} s_{n_0+n_1,\,n}=r_{n_0+n_1}$ gibt es in der Folge $s_{n_0+n_1,1}$, $s_{n_0+n_1,2}$ $\ldots s_{n_0+n_1,n}$ ein erstes Glied $s_{n_0+n_1,\,n_2}$ fuer das $s_{n_0-1}+s_{n_0,\,n_1-1}+s_{n_0+n_1,\,n_2}\geq x$ ist.

Dann ist:

$$s_{n_0-1}+s_{n_0,\,n_1-1}+s_{n_0+n_1,\,n_2-1}$$
$$<x\leq s_{n_0-1}+s_{n_0,\,n_1-1}+s_{n_0+n_1,\,n_2-1}+a_{n_0+n_1+n_2},$$

und somit nach Annahme auch:

$$s_{n_0-1}+s_{n_0,\,n_1-1}+s_{n_0+n_1,\,n_2-1}+r_{n_0+n_1+n_2}\geq x.$$

Gilt hierin das $=$-Zeichen, so haben wir in

$$s_{n_0-1}+s_{n_0,\,n_1-1}+s_{n_0+n_1,\,n_2-1}+r_{n_0+n_1+n_2}$$

eine Teilreihe von (2) vor uns, die gleich x ist, und die Behauptung ist bewiesen. Andernfalls schliesst man in derselben Weise weiter. Kommt man so nach k Schritten zu einer Gleichung:

$$(9)\quad s_{n_0-1}+s_{n_0,\,n_1-1}+s_{n_0+n_1,\,n_2-1}+\cdots$$
$$+s_{n_0+n_1+\ldots+n_{k-1},\,n_k-1}+r_{n_0+n_1+\ldots+n_k}=x$$

so ist die linke Seite eine Teilreihe von (2), die$=x$ ist, und die Behauptung ist bewiesen. Kommt man aber bei Fortsetzung des Verfahrens niemals auf eine Gleichung (9), so gilt fuer alle k:

$$s_{n_0-1}+s_{n_0,n_1-1}+\dots+s_{n_0+n_1+\dots+n_{k-1},n_k-1}<x$$

$$\leq s_{n_0-1}+s_{n_0,n_1-1}+\dots+s_{n_0+n_1+\dots n_{k-1},n_k-1}+a_{n_0+n_1+\dots+n_k}$$

und wegen (1) folgt daraus:

$$s_{n_0-1}+s_{n_0,n_1-1}+s_{n_0+n_1,n_2-1}+\dots+s_{n_0+n_1+\dots+n_{k-1},n_k-1}+\dots=x;$$

da hierin die linke Seite eine Teilreihe von (2) ist, ist Satz 1 bewiesen. Als Korullar erhalten wir daraus:

Ist $\sum\limits_{\nu\geq1} a_\nu$ *eine divergente Reihe aus positiven, gegen o konvergierenden Gliedern, so gibt es in ihr zu jeder pesitiven Zahl* x *eine Teilreihe, fuer die* $\sum\limits_{i\geq1} a_{\nu_i}=x$ *ist.*

Denn da $r_n=+\infty$ ist fuer alle n, ist die Bedingung von Satz I erfuellt, und hier ist $g=+\infty$.

Wenn in der Reihe (2) die Glieder a_ν monoton abnehmen $(a_\nu\geq a_{\nu+1})$, so ist die in Satz I als hinreichend erwiesene Bedingung auch notwendig:

II. *Sei* (2) *eine Reihe aus positiven, monoton abnehmenden Gliedern: damit es zu jeder der Ungleichung* $0<x\leq g$ *genuegenden Zahl* x *eine Teilreihe von* (2) *gebe, fuer die* $\sum\limits_{i\geq1} a_{\nu_i}=x$ *ist, ist notwendig, dass fuer alle* $n\geq1$ *gelte:* $r_n\geq a_n$.

Angenommen, es gebe ein $n^*\geq1$, so dass $r_{n^*}<a_{n^*}$. Wegen

$$s_{n^*}=s_{n^*-1}+a_{n^*} \text{ ist dann } s_{n^*-1}+r_{n^*}<s_{n^*}.$$

Man waehle nun x gemaess der Ungleichung:

$$s_{n^*-1}+r_{n^*}<x\leq s_{n^*}. \tag{10}$$

Sei sodann $\sum\limits_{i\geq1} a_{\nu_i}$ eine beliebige Teilreihe von (2). Kommen $a_1, a_2,\dots a_{n^*}$ unter den a_{ν_i} vor, so ist $\sum\limits_{i\geq1} a_{\nu_i}>s_{n^*}$, also wegen (10) auch $\sum\limits_{i\geq1} a_{\nu_i}>x$.

Kommt hingegen eines der Glieder $a_1, a_2, \dots a_{n^*}$ unter den a_{ν_i} nicht vor, so ist, da die a_ν monoton abnehmen und somit das fehlende Glied $\geq a_{n^*}$ ist: $\sum\limits_{i\geq 1} a_{\nu_i} \leq s_{n^*-1} + r_{n^*}$, also wegen (10) auch: $\sum\limits_{i\geq 1} a_{\nu_i} < x$. Jedenfalls ist also $\sum\limits_{i\geq 1} a_{\nu_i} \neq x$, es kann somit keine Teilreihe von (2) eine der Ungleichung (10) genuegende Zahl darstellen.

§ 2

Wir betrachten wieder die Reihe (2) aus positiven, monoton abnehmenden, der Bedingung (1) genuegenden Gliedern; wir nehmen an, die Bedingung $r_n \geq a_n$ von Satz I und II sei erfuellt und fragen: Unter welchen Umstaenden gibt es zu jeder der Ungleichung $0 < x \leq g$ genuegenden Zahl x nur eine einzige Teilreihe von (2), fuer die

$$\sum_{i\geq 1} a_{\nu_i} = x \text{ ist ?}$$

Dabei wollen wir, wenn unter den Gliedern a_ν einander Gleiche vorkommen, die beiden Teilreihen $\sum\limits_{i\geq 1} a_{\nu_i}$ und $\sum\limits_{i\geq 1} a_{\mu_i}$ als *dieselbe* Teilreihe von (2) betrachten, wenn $a_{\nu_i} = a_{\mu_i}$ ist fuer alle i:

III. *Sei (2) eine Reihe aus positiven, monoton abnehmenden, der Bedingung (1) genuegenden Gliedern, in der $r_n \geq a_n$ sei fuer alle $n \geq 1$; damit es zu jeder der Ungleichung $0 < x \geq g$ genuegenden Zahl x nur eine einzige Teilreihe von (2) gebe, fuer die $\sum\limits_{i\geq 1} a_{\nu_i} = x$ ist, ist notwendig und hinreichend, dass fuer alle, fuer die $a_{n+1} < a_n$ ist $r_n = a_n$ sei.*

Notwendig: Sei:

$$a_{n^*+1} < a_{n^*}, \; r_{n^*} > a_{n^*}. \tag{11}$$

Wir setzen:

$$x^* = s_{n^*-1} + r_{n^*}; \tag{12}$$

wegen $s_{n^*} = s_{n^*-1} + a_{n^*}$ folgt aus (11): $x^* > s_{n^*}$ und aus (12) folgt, bei Beachtung von (11):

$$x^* - s_{n^*} = s_{n^*-1} - s_{n^*} + r_{n^*} = r_{n^*} - a_{n^*} < r_{n^*} - a_{n^*+1} = r_{n^*+1}.$$

Wir haben also:

$$0 < x^* - s_{n^*} < r_{n^*+1}.$$

Nach (3) ist aber $r_{n^*+1} = \sum_{\nu \geq 1} a_{n^*+1+\nu}$ eine unendliche Reihe aus positiven, monoton abnehmenden Gliedern, die der Bedingung (1) genuegen; und nach Satz I gibt es eine Teilreihe von r_{n^*+1}, sodass

$$x^* - s_{n^*} = \sum_{i \geq 1} a_{n^*+1+\nu_i} \tag{13}$$

Nach (12) und (13) haben wir also fuer x^* die beiden Darstellungen:

$$x^* = s_{n^*-1} + \sum_{\nu \geq 1} a_{n^*+\nu} \quad ; \quad x^* = s_{n^*} + \sum_{i \geq 1} a_{n^*+1+\nu_i} \tag{14}$$

In beiden Darstellungen ist die rechte Seite eine Teilreihe von (2) und zwar sind dies zwei verschiedene Teilreihen von (2); denn alle Glieder von s_{n^*-1} kommen in beiden vor; auf diese n^*-1 gemeinsamen Glieder aber folgt in der ersten Teilreihe das Glied a_{n^*+1}, in der zweiten aber das Glied a_{n^*}, und nach (11) ist $a_{n^*} \neq a_{n^*+1}$. Es gibt also in (2) zwei verschiedene die Zahl x^* darstellende Teilreihen.

Hinreichend:

Seien $\sum_{i \geq 1} a_{\nu_i}$ und $\sum_{i \geq 1} a_{\mu_i}$ zwei verschiedene Teilreihen von (2) und es sei i^* der kleinste Index, fuer den $a_{\nu_{i^*}} \neq a_{\mu_{i^*}}$; wir koennen ohne weiteres annehmen, es sei $a_{\nu_{i^*}} < a_{\mu_{i^*}}$, also wegen der Monotonie der a_ν:

$$\nu_{i^*} > \mu_{i^*} \tag{15}$$

Wir setzen:

$$\sum_{1 \leq i < i^*} a_{\nu_i} = \sum_{1 \leq i < i^*} a_{\mu_i} = \sigma$$

(wenn $i^* = 1$ wird $\sigma = 0$ gesetzt). Bezeichnen wir mit a_{μ^*} das letzte Glied von (2), das $= a_{\mu_{i^*}}$ ist, so ist wegen (15):

$$\mu^* < \nu_{i^*} \tag{16}$$

und da wir die Bedingung von Satz III als erfuellt voraussetzen, ist:

$$a_{\mu^*} = r_{\mu^*} \tag{17}$$

Ferner ist offenbar:

$$(18) \qquad \sum_{\iota \geq 1} a_{\nu_\iota} \leq \sigma + r_{\nu_{\iota^*-1}} \; ; \; \sum_{\iota \geq 1} a_{\mu_\iota} > \sigma + a_{\mu_{\iota^*}} \; (=\sigma + a_{\mu^*})$$

Wegen (16) ist hierin $\mu^* \leq \nu_{\iota^*-1}$ also bei Beachtung von (17):

$$r_{\nu_{\iota^*-1}} \leq r_{\mu^*} = a_{\mu^*}$$

so dass aus (18) folgt:

$$\sum_{\iota \geq 1} a_{\nu_\iota} < \sum_{\iota \geq 1} a_{\mu_\iota} .$$

Zwei verschiedene Teilreihen von (2) koennen also nicht dieselbe Zahl darstellen, und die Behauptung ist bewiesen.

Es ist nun leicht, alle moeglichen Reihen (2) aus positiven Gliedern anzugeben, durch deren saemtliche Teilreihen die saemtlichen der Ungleichung $0 < x \leq 1$ genuegenden Zahlen x dargestellt werden und zwar so, dass jeder solche x nur durch eine einzige Teilreihe dargestellt wird. Da es auf die Reihenfolge der Glieder nicht ankommt, koennen wir wieder die als monoton abnehmend voraussetzen. Jedenfalls muss in (2) $g=1$ sein; denn waere $g<1$, so waere keine der Ungleichung $g<x\leq 1$ genuegende Zahl x durch eine Teilreihe von (2) darzustellen, und waere $g>1$, so wuerde die Reihe (2) selbst, die ja auch eine ihrer Teilreihen ist, eine nicht der Ungleichung $0<x\leq 1$ genuegende Zahl darstellen. Die Reihe (2) muss also konvergent sein, so dass (I) von selbst erfuellt ist. Sei etwa $a_1 = a_2 = \ldots = a_{n_1} > a_{n_1+1}$. Dann muss nach III $r_{n_1} = a_{n_1}$ sein; aus:

$$\sum_{\nu \geq 1} a_\nu = a_1 + a_2 + \ldots + a_{n_1} + r_{n_1} = 1$$

folgt also:

$$a_1 = a_2 = \ldots = a_{n_1} = \frac{1}{n_1+1} \; ; \; r_{n_1} = \frac{1}{n_1+1} .$$

Sei sodann:

$$a_{n_1} = a_{n_1+2} = \ldots = a_{n_1+n_2} > a_{n_1+n_2+1} \; ;$$

nach III muss $r_{n_1+n_2} = a_{n_1+n_2}$ sein, und aus:

$$a_{n_1+1} + a_{n_1+2} + \ldots + a_{n_1+n_2} + r_{n_1+n_2} = r_{n_1} \left(= \frac{1}{n_1+1} \right)$$

folgt:

$$a_{n_1+1}+a_{n_1+2}+\dots+a_{n_1+n_2}$$

$$=\frac{1}{(n_1+1)(n_2+1)}\,;\; r_{n_1+n_2}=\frac{1}{(n_1+1)(n_2+1)}.$$

Indem man so fortschliesst, sieht man, dass die Reihe folgende Gestalt haben muss:

Es gibt eine Folge natuerlicher Zahlen $n_1, n_2, \dots n_k$ so dass:

$$a_1=a_2=\dots=a_{n_1}=\frac{1}{n_1+1}\,;\; a_{n_1+1}=a_{n_1+2}=\dots=a_{n_1+n_2}=\frac{1}{(n_1+1)(n_2+1)}.$$

Diese Reihen sind bekannt als *Cantorsche Reihen* (vergl. z.B.O. Perron, *Irrationalzahlen*, p. 111), und wir haben den Satz bewiesen:

IV. *Die Cantorschen Reihen sind die einzigen Reihen aus positiven Gliedern, deren saemtliche Teilreihen die saemtlichen der Ungleichung* $0<x\leq 1$ *genuegenden Zahlen so darstellen, dass jede solche Zahl nur durch eine einzige Teilreihe dargestellt wird.*

Die Systembrüche der Grundzahl $g(>1)$ sind in den Cantorschen Reihen als der Spezialfall $n_k=g-1$ $(k=1, 2, \dots)$ enthalten.

§ 3

Wir verwenden nun Satz I zum Beweise des in der Einleitung erwaehnten Zwischenwertsatzes in der Theorie der absolut additiven Mengenfunktionen. Wegen aller im folgenden verwendeten Begriffe und Saetze aus der Theorie der absolut additiven Mengenfunktionen sei verwiesen auf H. Hahn, *Theorie der reellen Funktionen*, sechstes Kap.

Sei $\phi(M)$ eine im σ-Koerper M definierte absolut additive Mengenfunktion. Alle weiterhin auftretenden Mengen, gehoeren, auch wenn dies nicht ausdruecklich gesagt wird, zum σ-Koerper M. Eine Menge S heisst *singulaer* fuer ϕ, wenn $\phi(S)\neq 0$ ist und fuer jeden Teil J vom S entweder $\phi(J)=0$ oder $\phi(J)=\phi(S)$ ist. Besitzt die Menge A keinen fuer ϕ singulaeren Teil, so heisse ϕ *singularitaetenfrei* in A.

V. *Ist* $\phi(M)$ *singularitaetenfrei* in A, *so auch die Positivfunktion* $\pi(M)$ *und die Negativfunktion* $\nu(M)$ *von* ϕ. Wir beweisen dies etwa fuer $\pi(M)$. Angenommen, es gaebe in A einen fuer $\pi(M)$ singulaeren Teil P. Wir zerlegen (l.c. meine *Theorie*, p. 404, Satz IX.) P in zwei fremde Teile $P=P'+P''$, so dass $\pi(P')=\pi(P)$, $\pi(P'')=0$, $\nu(P')=0$, $\nu(P'')=\nu(P)$.

Fuer jeden Teil M von P′ gilt dann $\phi(M)=\pi(M)$. Da P singulaer fuer $\pi(M)$, ist $\pi(P)\neq 0$, also auch $\pi(P')\neq 0$, also auch $\phi(P')\neq 0$: und fuer jeden Teil M von P′ gilt $\pi(M)=0$ oder $\pi(M)=\pi(P)=\pi(P')$ also auch $\phi(M)=0$ oder $\phi(M)=\phi(P')$; d.h. P′ ist singulaer fuer ϕ. Gibt es also in A einen fuer π singulaeren Teil P, so gibt es in A auch einen fuer ϕ singulaeren Teil P′. Damit ist die Behauptung bewiesen.

Wir nennen die absolut additiv Mengenfunktion $\phi(M)$ monoton wachsend in A, wenn $\phi(M)\geq 0$ ist fuer alle Teile von A.

VI. *Ist ϕ in A monoton wachsend und singularitaetenfrei, und ist* $\phi(A)>0$, *so gibt es in A eine Folge von Teilen* $B_1, B_2,\ldots B_{n_1},\ldots$ *so dass* $\phi(B_n)>0$ *fuer alle n und* $\lim\limits_{n\to\infty}\phi(B_n)=0$. Weil ϕ in A singularitaetenfrei, gibt es einen Teil A von A_1, so dass $0<\phi(A_1)<\phi(A)$, ebenso einen Teil A_2 von A, so dass $0<\phi(A_2)<\phi(A_1)$ usw. Man erhielt so eine Folge von Teilen A_1, $A_2,\ldots A_r$ von A…so dass $\phi(A_1)>\phi(A_2)>\ldots>\phi(A_n)>\ldots>0$.

Es existiert also $\lim\limits_{n\to\infty}\phi(A_n)$. Setzen wir nun $A_n-A_{n+1}=B_n$, so ist $\phi(B_n)=\phi(A_n)-\phi(A_{n+1})$, und somit $\phi(B_n)>0$ und $\lim\limits_{n\to\infty}\phi(B_n)=0$.

VII. *Ist ϕ in A monoton wachsend und singularitaetenfrei und ist* $\phi(A)=+\infty$ *so gibt es zu jeder noch so grossen Zahl z einen Teil B von A, so dass* $z<\phi(B)<+\infty$. Da ϕ singularitaetenfrei in A, gibt es jedenfalls Teile von M, fuer die $0<\phi(M)<+\infty$. Bilden wir fuer jeden solchen Teil von M den Funktionswert $\phi(M)$, so hat die Menge aller dieser $\phi(M)$ eine obere Grenze g. Unsere Behauptung ist gleichbedeutend mit: $g=+\infty$. Angenommen, es waere $g<+\infty$. Sicherlich gibt es in A eine Folge von Teilen $M_1, M_2,\ldots M_{n_1},\ldots$ so dass $\lim\limits_{n\to\infty}\phi(M_n)=g$. Bezeichnen wir die Vereinigung von $M_1, M_2,\ldots M_n$, mit A_n so ist: $\phi(A_n)\leq\phi(M_1)+\phi(M_2)+\ldots\phi(M_n)$ also ist auch $\phi(A_n)$ endlich, mithin $\phi(A_n)\leq g$; und da $\phi(A_n)\geq\phi(M_n)$ folgt aus $\lim\limits_{n\to\infty}\phi(M_n)=g$ auch $\lim\limits_{n\to\infty}\phi(A_n)=g$. Sei nun B die Vereinigung von $A_1, A_2\ldots A_n,\ldots$ Da Die Mengenfolge $A_1, A_2,\cdots A_n,\ldots$ monoton waechst, ist dann bekanntlich $\phi(B)=\lim\limits_{n\to\infty}\phi(A_n)$ also $\phi(B)=g$. Da $g<+\infty$ und $\phi(A)=+\infty$, Ist auch $\phi(A-B)=+\infty$. Da ϕ singularitaetenfrei in A gibt es in A−B einen Teil ϕ, so dass $0<\phi(\phi)<+\infty$. Da B und ϕ 1st $\phi(B+\phi)=\phi(B)+\phi(\phi)$ $=g+\phi(\phi)$, also $g<\phi(B+\phi)<+\infty$; das aber widerspricht der Definition von g als der oberen Grenze aller fuer Teile M von A auftretenden endlichen Funktionswerte Die Annahme $g<+\infty$ fuehrt also auf einen Widerspruch, und die Behauptung ist bewiesen.

Nach diesen Vorbereitungen kommen wir nun zum Beweise des Zwischenwertsatzes. Bekanntlich (l. c. meine *Theorie*, p. 401, Satz III, IV) gibt es unter allen Werten $\phi(M)$ die ϕ fuer Teile M von A annimmt, sowohl einen groessten als einen kleinsten und zwar sind dieser groesste und kleinste Wert gegeben durch π (A) und $-\nu(A)$. Der Zwischenwertsatz kann so ausgesprochen werden:

VIII. *Ist ϕ singularitaetenfrei in A, so gibt es zu jeder der Ungleichung $-\nu(A)\leq z\leq\pi(A)$ genuegenden Zahl z einen Teil M von* A *so dass* $\phi(M)=z$.

Beim Beweise koennen wir uns auf den Fall beschraenken, dass ϕ in A monoton wachsend ist. Denn nehmen wir an, der Satz gelte fuer monoton wachsende Mengenfunktionen; da $\pi(M)$ und ν (M) monoton wachsend und zufolge V in A singularitaetenfrei sind, kann dann der Satz auf π (M) und $\nu(M)$ angewendet werden; wir zerlegen nun A in zwei fremde Teile: $A=A'+A''$ so dass $\pi(A')=\pi(A)$, $\pi(A'')=0$, $\nu(A')=0$, $\nu(A'')=\nu(A)$ fuer jeden Teil M von A' ist dann π $(M)=\phi(M)$ und fuer jeden Teil M von A'' ist ν $(M)=-\phi(M)$; durch Anwendung unseres Satzes auf π und ν folgt dann: ist $0\leq z\leq\pi(A')$ $(=\pi(A))$ so gibt es einen Teil M von A', so dass $\pi(M)=z$ und mithin auch $\phi(M)=z$; ist $0\geq z\geq-\nu$ $(A'')(=-\nu(A))$ so gibt es einen Teil M von A'', so dass $-\nu(M)=z$ und mithin auch ϕ $(M)=z$; damit ist dann die Behauptung von Satz VIII fuer beliebiges ϕ bewiesen. Nehmen wir also nun $\phi(M)$ als monoton wachsendan, dann ist $\pi(M)=\phi(M)$, $\nu(M)=0$. Wir koennen weiter voraussetzen: $\phi(A)<+\infty$; denn nehmen wir an, der Satz sei bewiesen, wenn $\phi(A)$ endlich; ist nun ϕ $(A)=+\infty$, so gibt es nach Satz VII zu jedem $z\geq0$ einen Teil B von A, so dass $\phi(B)<z$; nach Annahme kann nun der Satz statt auf A auf B angewendet werden, und ergibt die Existenz eines Teiles M von B, fuer den $\phi(M)=z$; da aber M auch Teil von A ist, ist damit auch die Behauptung fuer A bewiesen. Wir haben also nur mehr zu zeigen: Ist $\phi(M)$ in A monoton wachsend, und $\phi(A)<+\infty$ so gibt es zu jedem der Ungleichung $0\leq z\leq\phi(A)$ genuegenden Zahl z einen Teil M von A, so dass $\phi(M)=z$. Fuer $z=0$ ist die Behauptung trivial; denn fuer die leere Menge L gilt: $\phi(L)=0$. Wir nehmen also $z>0$ an. Wir betrachten alle diejenigen Teile M_1 von A, fuer die $0<\phi(M_1)\leq\frac{1}{2}\phi(A)$; nach VI gibt es solche Teile M_1; die obere Grenze der Werte $\phi(M_1)$. die auf diesen Teilen M_1 von A annimmt, bezeichnen wir mit g_1: dann ist

$$0<g_1\leq\tfrac{1}{2}\,\phi(A), \tag{19}$$

und es gibt gewiss einen Teil A_1 von A, so dass

$$\tfrac{1}{2}\,g_1<\phi(A_1)\leq g_1 \tag{20}$$

Aus (19) und (20) folgt $\phi(A_1)<\phi(A)$ also $\phi(A-A_1)>0$. Nun betrachten wir alle Teile M_2 von $A-A_1$, fuer die $0<\phi(M_2)\leq\frac{1}{2}\phi(A-A_1)$ die obige Grenze aller auf diesen Teilen M_2 auftretenden Funktionswerten $\phi(M_2)$ bezeichenen wir mit g_2; dann ist

$$(21) \qquad 0<g_2\leq\tfrac{1}{2}\phi(A-A_1)$$

und es gibt gewiss einen Teil A_2 von $A-A_1$ so dass:

$$(22) \qquad \tfrac{1}{2}g_2<\phi(A_2)\leq g_2$$

Aus (21) und (22) folgt $\phi(A_2)<\phi(A-A_1)$ also $\phi(A-(A_1+A_2))>0$.

Nun betrachten wir alle Teile M_3 von $A-(A_1+A_2)$, fuer die $0<\phi(M_3)\leq \frac{1}{2}\phi(A-(A_1+A_2))$; die obere Grenze aller auf diesen Teilen M_3 auftretenden Funktionswerten bezeichenen wir mit g_3; dann ist:

$$0<g_3\leq\tfrac{1}{2}\phi(A-(A_1+A_2))$$

und es gibt einen Teil A_3 von $A-(A_1+A_2)$, so dass:

$$\tfrac{1}{2}g_3<\phi(A_3)\leq g_3.$$

In dieser Weise schliessen wir fort. Wir erhalten so eine Folge von Teilen $A_1, A_2, \ldots A_n, \ldots$ von A und eine Folge von Zahlen $g_1, g_2, \ldots g_n, \cdots$ mit folgenden Eigenschaften:

(1) A_n ist Teil von $A-(A_1+A_2+\ldots+A_{n-1})$;

(2) $0<g_n\leq\frac{1}{2}\phi(A-(A_1+A_2+\ldots+A_{n-1}))$;

(3) $\frac{1}{2}g_n<\phi(A_n)\leq g_n$.

Wegen (1) sind die Mengen $A_1, A_2, \cdots A_n, \ldots$ zu je zweien fremd, also ist, wenn

$$(23) \qquad A_1+A_2+\cdots+A_n+\ldots=B \text{ gesetzt wird,}$$

$$(24) \qquad \sum_{n\geq1}\phi(A_n)=\phi(B)\leq\phi(A)$$

und mithin: $\lim\limits_{n\to\infty}\phi(A_n)=0$.

Aus (3) folgt daher auch:

$$(25) \qquad \lim_{n\to\infty} g_n=0$$

Wir behaupten nun, dass fuer die Menge (23) gilt:

$$(26) \qquad \phi(A-B)=0 .$$

Denn waere $\phi(A-B)>0$, so gaebe es nach VI einen Teil ϕ von $(A-B)$ so dass

$$(27) \qquad 0<\phi(\phi)<\tfrac{1}{2}\,\phi\,(A-B).$$

Nach (23) ist ϕ Teil von $B-(A_1+A_2+\ldots+A_{n-1})$

also ist wegen (27): $\quad g_n \geq \phi(\phi)$,

was wegen $\phi(\phi)>0$ mit (25) in Widerspruch steht. Damit ist (26) nachgewiesen. Aus (26) folgt nun $\phi(A)=\phi(B)$, also wegen (23):

$$(28) \qquad \phi(A)=\sum_{\nu\geq 1}\phi(A_\nu)$$

Da ferner $A-(A_1+A_2+\ldots+A_{n-1})=A_n+A_{n+1}+\ldots+(A-B)$ folgt aus (26) auch:

$$(29) \qquad \sum_{\nu\geq n}\phi(A_\nu)=\phi(A-(A_1+\cdots+A_{n-1}) .$$

Wegen (2) und (3) ist hierein

$$(30) \qquad \phi\,(A_n)\leq\tfrac{1}{2}\,\phi(A-(A_1+\ldots+A_{n-1}))$$

Aus (29) und (30) aber folgt:

$$\sum_{\nu\geq n}\phi\,(A_\nu)\geq\phi\,(A_n).$$

Die Reihe (24) erfuellt also die Bedingung von Satz I, und aus Satz I folgt: ist $0<z\leq\phi(A)$, so gibt es in (24) eine Teilreihe, so dass $\sum_{i\geq 1}\phi(A_{\nu_i})=z$. Bezeichnen wir also mit M die Vereinigung von A_{ν_1}, A_{ν_2},...so ist $\phi(M)=z$, und Satz VIII ist bewiesen.

SOME REMARKS CONCERNING A RESULT OF BESICOVITCH

BY

AVADHESH NARAYAN SINGH (*Lucknow*)

[*Read July 7, 1929*]

1. The question whether there exists a continuous function which possesses at no point a progressive (regressive) derivative was first proposed by Professor Denjoy in a paper published in the *Journal de Mathematique* (7), Vol. 1 (1915).[1] An attempt to answer the above question in the affirmative has been made by A. Besicovitch, in a paper published in the *Bulletin of the Russian Academy of Sciences*, Vol. LXX, (1925), pp. 527-540, by the construction of an example of a continuous function which is stated to be devoid of the progressive as well as the regressive derivative everywhere.

The object of the present paper is to point out certain defects in the reasoning employed by Besicovitch for proving the non-existence of the derivative, and thus show that Besicovitch's work fails to answer the question proposed by Denjoy.[2]

2. Besicovitch's function is defined as follows: "Let us take the stretch AB$=2a$ for A $(0,0)$ and B $(2a, 0)$, and the points C (a, b) and D $(a, 0)$. On the stretch AD let us construct a stretch $l_1=\frac{a}{4}$ whilst we place it centrally. The stretch AD is divided by the stretch l_1 into two equal parts.

[1] *Cf.* Hahn: Über stetige Funktionen ohne Ableitung (*Jahresbericht d. deutsch. Math. Ver*, Vol. 26, 1918).

[2] Besicovitch's example has been recently considered by Miss E. D. Pepper in the *Fundamenta Mathematicae*, Vol. XII, pp. 244-258, where she has arrived at the same conclusions as Besicovitch. As the proof given by her does not differ in any essential from that given by Besicovitch, and as my criticism applies to both, I have not considered it necessary to deal with her proof separately.

On each of these let us place centrally the stretches $l_2 = l_3 = \frac{a}{2^4}$. The stretches l_1, l_2, l_3 divide the stretch AD into four equal stretches. On each of these let us place centrally (calculated from left to right) the stretches $l_4 = l_5 = l_6 = l_7 = \frac{a}{2^6}$, and so on. In this manner a set L of stretches

$$l_1 + l_2 + l_3 + \ldots\ldots = \frac{a}{2}$$

is constructed on the stretch AD.

We construct a similar system of stretches on DB. We call these stretches the first series of stretches.

Let us denote by $m(x)$ the measure [1] of the set of points of the interval $(0, x)$ which do not belong to the set L, and let us determine on the stretch AD a function $\phi(x)$, whilst we assume

$$\phi(x) = \frac{2b}{a}\, m(x)$$

The points A and D are thus connected by the curve $y = \phi(x)$, which has a constant value on an arbitrary stretch l_r, and which we call a 'ladder curve'; the points C and B are likewise connected by such a ladder curve. The figure originating in this manner is called a 'step-triangle' whose base is $2a$ and whose height is b (see Fig. 1).

On the fundamental lines corresponding to the first series of stretches of the step-triangle ABC, let us construct step-triangles directed towards below, equal on equal fundamental lines, whilst we choose the height so that the vertex of the undermost of all equal triangles lies on the side AB. The construction of all these triangles is called the operation of 'maiming' the triangle ABC towards inside. With the so obtained infinity of triangles (first series) we carry out the same operation of maiming towards inside, and thus obtain the second series of triangles; on them also perform maiming towards inside and so on.

We now define a function $f(x)$ on the stretch AB as follows:

(1) at the points of the stretch AB, which do not belong to the first series of stretches, by the ordinates of the sides of the step-triangle ABC,

[1] It has been assumed that the measure $m(x)$ exists as a unique number for every x.

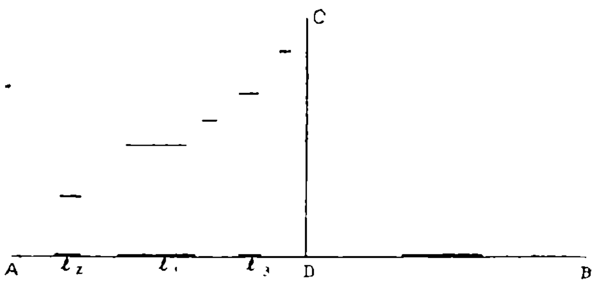


Fig 1

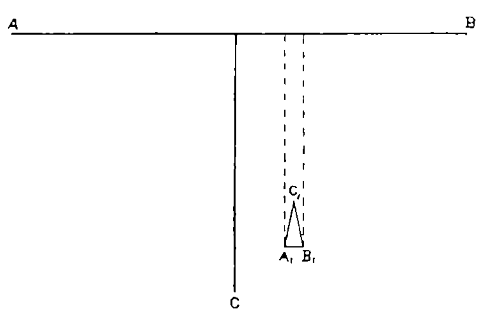


Fig. 2

(2) at the points of the stretches of the first series, which do not belong to the stretches of the second series, by the ordinates of the sides of the triangles of the first series,

(3) at the points of the stretches of the second series, which do not belong to the stretches of the third series, by the ordinates of the sides of the triangles of the second series, and so on,

(4) at the points which belong to the stretches of all series (they form a null set) according to the principle of continuity."

3. It must be pointed out that the definition of the function $f(x)$ depends on an infinite number of separate constructions. If we stop at any stage n, however great n may be, the function remains undefined, so that, in order to define $f(x)$ we have to make $n=\infty$, and therefore the construction can never be completed.

But, even if we suppose that the function is capable of definition in the manner indicated, the reasoning adopted by Besicovitch fails to prove the non-existence of the derivative at all the points of AB. Whilst Besicovitch gives the proof for the non-existence of the derivative at the points external to the stretches of a finite series in great detail, he disposes off the case of the points which lie within the stretches of all series in a few lines. The reasoning adopted by him is as follows:

"Let us take two step-triangles of two series following each other, of sufficient large index, which enclose the point d of the curve (see Fig. 2). Let ABC be the greater triangle, $A_1B_1C_1$ which is inside the first, the smaller triangle. Their vertices C and C_1 lie on different sides of their fundamental lines. Let B and B_1 be the right ends of the fundamental lines and let the triangle $A_1B_1C_1$ be situated in the right part of the triangle ABC. Finally let ϵ be an arbitrarily small positive number. It is now easy to convince oneself that for sufficiently high order of the triangle series, one has (see Fig. 2)

$$\angle B_1DB > \frac{\pi}{2} - \epsilon \text{ and } \angle CDA > \frac{\pi}{2} - \epsilon.$$

If we note, however, that the points A, A_1, B, B_1 belong to the curve $y=f(x)$, then we conclude that the oscillation of the ray which goes out from the given point of the curve $y=f(x)$ to the infinitely neighbouring points lying on both sides of the given point will exceed the angle $\frac{\pi}{2}-\epsilon$, where

ϵ is an arbitrarily small number. This establishes [1] the non-existence, at the point d of a right as well as also of a left derivative."

4. The reasoning given by Besicovitch does not establish the non-existence of the derivative at all the points of the set of the second category defined as interior points of the stretches L_1, L_2, L_3,......... The abscissa of the point d is supposed to be defined as the limit of a sequence of intervals, one within the other, tending to zero. In fact, this is essential for the proof of Besicovitch to hold. Thus his argument simply proves the non-existence of the derivative at points which are each supposed to lie within an infinite number of intervals, one within the other, tending to zero. The set of intervals at any stage is enumerable, so that, if one considers points d as has been done by Besicovitch, his reasoning will apply to an enumerable set of points only, and no more.

The above argument is made clearer by the consideration of the following example given by Borel : [2]

Let us suppose that each rational point $\frac{p}{q}$ in the interval (0, 1) is enclosed in the interval $\left(\frac{p}{q}-\frac{\lambda}{q^3}, \frac{p}{q}+\frac{\lambda}{q^3}\right)$, where λ has the same value for all the points. In this manner the rational points are enclosed in a set of overlapping intervals whose sum is less than $\lambda \sum (q-1) . \frac{2}{q^3}$, or than $2\lambda \sum \frac{1}{q^2}$, which can be made as small as we please by choosing λ small enough. The equivalent set of non-overlapping intervals defines, by means of its end points and their limiting points a closed set.

Now consider the set of points defined by

$$x=\frac{a_1}{10}+\frac{a_2}{10^{2!}}+\frac{a_3}{10^{3!}}+\ldots+\frac{a_n}{10^{n!}}+\ldots\ldots$$

[1] It may be true that when n is finite, intervals of the nth, and $(n+1)$th stages can be found, such that from d two secants different from each other can be drawn, but this property need not necessarily hold when $n=\infty$. We cannot visualise by means of a figure whether the two secants will remain different when n becomes infinite, for all points d.

[2] *Leçons sur la théorie des fonctions*, p. 44. Also see Hobson, *Theory of Functions of a Real Variable*, Vol. I (1927), p. 141.

where each $a \leq 9$, and the a's are such that an infinite number of them are different from zero. Let

$$\frac{p}{q}=\frac{a_1}{10}+\frac{a^2}{10^{2!}}+\ldots+\frac{a_n}{10^{n!}},$$

thus

$$q=10^{n!},$$

then

$$\left(x-\frac{p}{q}\right)=\frac{a_{n+1}}{10^{(n+1)!}}+\ldots\ldots<\frac{1}{q^n}\left(\frac{a_{n+1}}{q}+\ldots\right)<\frac{1}{q^n}.$$

It follows that, if x belongs to the above set, it is interior to the interval $\left(\frac{p}{q}-\frac{\lambda}{q^3}, \frac{p}{q}+\frac{\lambda}{q^3}\right)$; for suppose $q=10^{n!}$; then

$$\left|\frac{p}{q}-x\right|<\frac{1}{q^n}<\frac{\lambda}{q^3},$$

provided

$$\lambda \geq \frac{1}{10^{(n-3)n!}},$$

and however small λ may be, values of n can be found for which the inequality is satisfied.

It thus appears that, besides the original points $\frac{p}{q}$ which the intervals are drawn to enclose, there are other points which lie inside the intervals for all values of λ, when λ is diminished indefinitely.

When λ has the values of a diminishing sequence $\{\lambda_n\}$ tending to zero, the set of intervals $\left\{\left(\frac{p}{q}-\frac{\lambda}{q^3}, \frac{p}{q}+\frac{\lambda}{q^3}\right)\right\}$ defines the point $\frac{p}{q}$ only. Thus the original points $\frac{p}{q}$ are the only ones that are defined as the limit of a sequence of intervals of the above type. The other points x which lie within

the intervals for all values of λ are clearly incapable of definition in the above manner.

It is, therefore, clear that Besicovitch's reasoning fails to prove the non-existence of the derivative at all the points of AB.

NOTE ON THE STATICAL GRAVITATIONAL FIELD WITH AXIAL SYMMETRY

N. R. SEN AND N. N. GHOSH

[*Read December 15, 1929*]

Statical gravitational fields with axial symmetry have been the subject of investigation by many mathematicians and the most important results in this line are contained in the papers of Weyl,[1] Levi Civita[2] and R. Bach.[3] The pioneer worker is Weyl who has shown that in space free from matter as well as for certain special distributions of the material energy tensor, if the gravitational field possesses rotational symmetry, a set of canonical co-ordinates exists in terms of which the whole space can be represented (Abbildung) by a Euclidean space in r, θ, z co-ordinates. Levi Civita's results go a little further than those of Weyl while Bach has attempted a solution of the problem of two bodies possessing axial symmetry, and the gravitational field of a ring, in which Weyl's co-ordinates have been quite useful.

Here we have started with the entire material energy tensor in Einstein's equations and proceeded with the algebraic simplification as far as possible. It appears that for a given distribution of the covariant energy tensor, differential equations, generally non-linear, as is usual in Relativity, of the second order can be written down explicitly for all the components of the fundamental tensor and a general form of Levi Civita's expression which must necessarily be a perfect differential exists. Simplifying assumptions at successive stages lead to certain interesting particular solutions one of which gives constant light velocity in space occupied by matter. The cosmological conclusions from these results can also be easily understood from the point of view of classical mechanics.

2. A field with axial symmetry can be defined by the line-element

$$ds^2 = e^{2f} dx_4^2 - e^{2\lambda}(dx_1^2 + dx_2^2) - e^{2l} dx_3^2 \qquad \ldots \quad (1)$$

[1] Weyl, Ann. d. Physik, 54 (1917), 59 (1919).

[2] Levi Civita, Rend. Acc. d. Lincei, 1917-19.

[3] R. Bach, Math. Zeitschr., 18 (1922).

were x_3 is supposed to vary from 0 to 2π and f, l, h are all functions of x and x_2. The gravitational equations of Einstein are

$$G_{\mu\nu}=-8\pi(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T) \qquad (\mu, \nu=1, 2, 3, 4) \qquad \dots \quad (2)$$

The tensor $G_{\mu\nu}$ has only five of its components different from zero and these have been calculated by many writers. We here write down the equations

$$f+l=\log r.$$

$$\frac{1}{r}\Delta r=-8\pi e^{2k}(T^1_1+T^2_2)=8\pi(T_{11}+T_{22}), \qquad \dots \quad (A)$$

$$\Delta f+\frac{1}{r}(r_1 f_1+r_2 f_2)=4\pi e^{2k}(T^4_4-T^1_1-T^2_2-T^3_3), \qquad (B)$$

$$\frac{1}{r}\{r_{12}-r_1(f_2+h_2)-r_2(f_1+h_1)\}+2f_1f_2=8\pi e^{2k}T^1_2=-8\pi T_{12}, \qquad (C)$$

$$\Delta(f+h)+(f_1^2+f_2^2)=-8\pi e^{2k}T^3_3, \qquad (D)$$

$$\frac{r_1}{r}(f_1+h_1)-\frac{r_2}{r}(f_2+h_2)+f_2^2-f_1^2-\frac{r_{11}}{r}=8\pi e^{2k}T^2_2=-8\pi T_{22}, \qquad (E)$$

where $f_1=\dfrac{\partial f}{\partial x_1},\ f_2=\dfrac{\partial f}{\partial x_2}, f_{11}=\dfrac{\partial^2 f}{\partial x_1^2},\ \Delta f=f_{11}+f_{22}=\dfrac{\partial^2 f}{\partial x_1^2}+\dfrac{\partial^2 f}{\partial x_2^2}$

etc.

Here we have 5 equations among 3+5 quantities so that any three of the unknown quantities can be chosen at random. If the three covariant stress-components in a " meridian " plane, *viz.*, T_{11}, T_{22} and T_{12} are given the differential equations satisfied by r, f, and h explicitly can all be written down. We first introduce the notation

$$\left.\begin{aligned} &\frac{1}{r}(r_1^2+r_2^2)=L, \\ &\frac{r_{12}}{r}+2f_1f_2+8\pi T_{12}=M, \\ &f_2^2-f_1^2-\frac{r_{11}}{r}+8\pi T_{22}=N, \end{aligned}\right\} \qquad \dots \quad (3)$$

and observe that equation (A) first determines r in terms of the stress components T_{11} and T_{22}. Then L, M, N contain only one unknown quantity namely f. With the help of the notation introduced we write (C) and (E) as follows

$$L(h_2+f_2)=Mr_1+Nr_2 \qquad \dots \quad (C')$$

$$L(h_1+f_1)=Mr_2-Nr_1. \qquad \dots \quad (E')$$

We thus see that the expression

$$\frac{1}{L}(Mr_2-Nr_1)dx_1+\frac{1}{L}(Mr_1+Nr_2)dx_2 \qquad \dots \quad (4)$$

which now contains only f as unknown quantity, *must be a perfect differential.* This is probably the most general form of Levi Civita's perfect differential expression. The elimination of h_2 from (C′) and (E′) by a single differentiation now gives a non-linear differential equation of the second order in f in terms of the known stress components and the function r which is also assumed to be known as a solution of (A). The mass density and "azimuthal tension" are then given by (B) and (D). To obtain the equation, for instance, for f we differentiate (C′) and (E′) with respect to both x_1 and x_2 and making use of (B), (D) and (3) get the equations

$$L\frac{r_2}{r}\Delta r_1-2L(r_1f_2-r_2f_1)4\pi e^{2\lambda}(T_4^4-T_1^1-T_2^2-T_3^3)=8\pi(T_{11}+T_{22})$$

$$\times\{M(r_2^2-r_1^2)-2Nr_1r_2\}+8\pi\frac{L}{r}(r_1^2-r_2^2)T_{12}+\frac{16\pi}{r}Lr_1r_2T_{22}$$

$$+8\pi L\{r_1(T_{12})_1-r_2(T_{12})_2\}+8\pi L\{r_1(T_{22})_2+r_2(T_{22})_1\} \qquad \dots \quad (C'')$$

and

$$-\frac{L}{r}r_1\Delta r_1-2L(r_1f_1+r_2f_2)4\pi e^{2\lambda}(T_4^4-T_1^1-T_2^2-T_3^3)-8\pi L^2e^{2\lambda}T_3^3$$

$$=8\pi(T_{11}+T_{22})\{N(r_1^2-r_2^2)-2Mr_1r_2\}-8\pi\frac{L}{r}(r_1^2-r_2^2)T_{22}+16\frac{\pi L}{r}$$

$$r_1r_2T_{12}+8\pi L\{r_1(T_{12})_2+r_2(T_{12})_1\}+8\pi L\{r_2(T_{22})_2-r_1(T_{22})_1\}. \qquad \dots \quad (E'')$$

In the second term on the left-hand side the factor containing $e^{2\lambda}$ is to be replaced by the equivalent expression from (B) if the equation in f is required. We prefer the present form for some special purpose.

3. We shall now discuss some particular solutions. If $T_1^1+T_2^2=0$ we have from (A)

$$\Delta r=0$$

which is Weyl's condition for the existence of canonical co-ordinates. In the more general case the equation is of the form

$$\Delta r = f(x_1, x_2).r$$

which may be compared with the equation of stationary vibration in two dimensions in a medium in which the velocity of propagation varies from point to point. A vibrating membrane with non-homogeneous distribution of mass also satisfies an equation of this type. It is necessary that the function $f(x, y)$ should be negative in such a case.

If, however, in any system the stresses in a meridian plane vanish so that

$$T_1^1 = T_2^2 = T_2^1 = 0$$

then also $$\Delta r = 0.$$

Equations (C″) and (E′′) are then reduced to the simple forms

$$\left.\begin{aligned} &L\frac{\partial(r, f)}{\partial(x_1, x_2)} \quad (T_4^4 - T_3^3) = 0 \\ \text{and} \quad &LT_3^3 + (r_1 f_1 + r_2 f_2).(T_4^4 - T_3^3) = 0. \end{aligned}\right\} \quad \ldots \quad (5)$$

These equations we can satisfy in various ways and the alternatives are given below.

(*a*) $T_4^4 - T_3^3 = 0$, $L \neq 0$. These are compatible with the second equation of (5) only when $T_4^4 = T_3^3 = 0$. The energy tensor disappears completely and the space is empty.

(*b*) Let $\frac{\partial(r, f)}{\partial(x_1, x_2)} = 0$. This means that f is a function of r so that there must also be such a relation as

$$l = \phi(f).$$

We shall see in the next article that the field in this case is one with the symmetry of a circular cylinder about the axis.

(*c*) A third possibility of satisfying the first of the above equations is to put f=const. From the second equation and (C), (D), (E) we have, when $L \neq 0$, *i.e.*, $r \neq$const in addition to two equations in r

$$T_3^3 = T_4^4 = 0. \quad \Delta h = 0.$$

The space is empty and it can also be verified by the calculation of Riemann Christoffel tensor that it is also Euclidean.

(*d*) Another interesting case arises when r=const. Weyl's canonical co-ordinates (r, z, θ) do not now exist. From (B), (C), (D) and (E) we have

$$f=\text{const}, \quad T_4^4-T_3^3=0, \quad \Delta h=-8\pi e^{2k}T_4^4=-4\pi\rho, \ \rho=2\pi e^{2k}T_4^4.$$

Unless the field is Euclidean, in which case $T_4^4=0$, we have a Poisson's equation in two-dimensional space x_1, x_2. Hence corresponding to any solution of Poisson's equation we shall have a gravitational field possessing an axis and the mass distribution in the two spaces will correspond with each other. But the interesting point in this connection is that here we have solutions of Einstein's equation giving constant velocity of light in material media ($T_4^4 \neq 0$). This result also admits of generalisation. If instead of assuming that all the stress components in a meridian plane vanish we take only $T_1^1+T_2^2=0$ and put

$$\rho_1=e^{2k}(T_3^3-T_4^4), \quad \rho_2=e^{2k}(T_3^3+T_4^4)$$

we get the equations of Weyl

$$\Delta f=-4\pi\rho_1, \qquad \Delta h+(f_1^2+f_2^2)=-4\pi\rho_2$$

though not in Weyl's canonical co-ordinates. There is nothing in the above investigation restricting x_3 to lie between 0 and 2π. It may be permissible to use the line element (1) for some cylindrical field having x_3 as axis and possessing the same property in all (x_1, x_2) " planes " or in other words the gravitational field of some infinite cylinder with x_3 as axis. We have just seen that outside such a cylindrical distribution of matter we can have (corresponding to r=const.) Euclidean space with constant light velocity. If, moreover, it may be arranged that the " stresses " in the (x_1, x_2) planes also vanish then the light velocity is also constant inside the cylinder. This furnishes a simple case showing that constant light velocity in the presence of matter is not incompatible with Einstein's equation.

We summarise these results as follows. When the gravitational field is defined by (1), the absence of " stresses " in (x_1, x_2) " planes " implies the total absence of matter except when the field has the symmetry of a circular cylinder or when Weyl's canonical (cylindrical) co-ordinates (r, z, θ) do not exist. In these exceptional cases l and f are connected together by some functional relation or are constants. If in addition the " azimuthal stress " T_3^3 also vanishes the space must needs be empty (without exception). It is not thus possible for a mass of incoherent dust particles of *any density* to remain in statical equilibrium in any form round an axis of symmetry. A spherical distribution of such particles is also consequently excluded. This result is quite easily understood from the point of view of classical mechanics. If we venture to apply it to cosmic problems we may say that a universe consisting of stars with any density distribution in space but possessing

rotational (or cylindrical) symmetry about an axis without internal relative motion or rotation cannot exist.

In a "cylindrical" distribution of matter with only azimuthal stresses the light velocity is the same constant inside and outside and the space outside is Euclidean.

4. By passage to Weyl's canonical co-ordinates some of the results of the previous article assume their well-known simple forms. Putting $T_1^1+T_2^2=0$ we obtain

$$L=\frac{1}{r},\quad M=2f_1f_2-8\pi e^{2k}T_2^1,\quad N=f_2^2-f_1^2-8\pi e^{2k}T_2^2$$

and we easily get to Weyl's equations in canonical co-ordinates. The light velocity f does not even then possess additive character except in the special case when $T_3^3=T_4^4$, f being then a potential function.

Equations (5) have now the simple forms

$$\left.\begin{aligned} &f_2(T_4^4-T_3^3)=0\\ &rf_1(T_4^4-T_3^3)+T_3^3=0\end{aligned}\right\}\qquad \dots\quad (5')$$

which can also be easily obtained by differentiating (C) and (E), both with respect to r and z and simplifying further with the help of (D) and (B).

We must have either $T_4^4=T_3^3=0$ so that the space is empty, or $f_2=0$ in which case f is a function of r only. The other equations are now

$$f_{11}+\frac{1}{r}f_1=4\pi e^{2k}(T_4^4-T_3^3),\quad h_2=0$$

$$f_1+h_1=rf_1^2,\quad (h_{11}+f_{11})+f_1^2=-8\pi e^{2k}T_3^3.$$

Here h and consequently T_3^3 and T_4^4 are all functions of r only. The field has the symmetry of a circular cylinder and the equations can be further integrated.

If also $T_3^3=0$ then T_4^4 must also vanish and we have the solutions [1]

$$f=\log\beta r^{a},\quad h=\log\delta r^{a(a-1)}$$

appropriate for a circular cylinder.

The condition that there should be no singularity on the axis or Euclidean geometry should ultimately exist at infinite distances from the axis both require a to be zero so that the space is Euclidean.

The expression (4) now reduces to Levi Civita's perfect differential expression

$$r\{(f_1^2+f_2^2-8\pi T_{12})dr+(2f_1f_2+8\pi T_{12})dz\}.$$

[1] G. Beck, Zeit. f. Physk., 33 (1925).

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: A SERIES OF COSECANTS.

BY

G. H. HARDY (OXFORD), AND J. E. LITTLEWOOD (CAMBRIDGE)

[*Read December 29, 1928.*]

1. We consider here some properties of the series

$$(1\cdot1) \qquad (S)\ \Sigma\ \frac{1}{n\sin n\theta\pi},\ (T)\ \Sigma\ \frac{1}{\sin n\theta\pi},$$

where θ is irrational. The analytical and arithmetical peculiarities of these series resemble those of the series

$$\Sigma\, e^{n^2\theta\pi i},\ \Sigma\,(n\theta-[n\theta]-\tfrac{1}{2}),^{1}$$

which have been discussed very thoroughly both by ourselves and by others.[2] The analysis connected with the series (1·1) is however in some ways particularly simple, and, although many writers, ourselves included, have considered similar series from time to time, there are some obvious questions concerning them which have remained unanswered.

We do not aim primarily at generality. The case in which we shall be interested particularly is that in which

$$\theta=a+\frac{1}{2a+}\,\frac{1}{2a+\dots}=\sqrt{(a^2+1)},$$

where a is an odd integer. It will however sometimes be obvious that our arguments apply to wider classes of irrationals. In particular a good many of our results hold for the class Θ of irrationals whose continued fractions have bounded co-efficients.

[1] $[x]$ is the integral part of x.

[2] Especially Hecke, Ostrowski, and Behnke. See the list of papers at the end.

We denote the general term of S by s_n, and the sum of its first n terms by S_n, and use a similar notation in T and other letters. It is plain that there is no universal upper bound for S_n, valid for all irrational θ; $|S_n|$ will be, for appropriate θ and n, larger than any assigned function $\phi(n)$. It is also plain that S cannot converge for any θ, since $|\sin n\theta\pi| < A/n$[1] for any θ and an infinity of n. The first question which suggests itself is whether S_n is *bounded* for any θ, and, if so, whether S is then summable by Cesàro's or other means. Our first object is to answer this question by proving Theorem I below.

Theorem I. *There are quadratic θ for which S has the following properties:*

(i) S *oscillates finitely*;

(ii) S *is not summable by any Cesàro mean*;

(iii) S *is summable by Riesz's logarithmic means of any positive order.*

In particular all this is true when

$$\theta=\sqrt{(a^2+1)}$$

where a is an odd integer, and the (Riessian) sum of the series is then

$$-\frac{\pi}{12}\sqrt{(a^2+1)}.$$

The theorem shews that the behaviour of the series in these respects is like that of

$$1-1+0+1+0+0+0-1+0+\dots$$

(where the ranks of the non-zero terms are 1, 2, 4, 8,...). The most difficult part of the proof is the proof of (i), which we defer to § 3. In § 2 we prove the rest of the theorem, taking (i) for granted where it is necessary.

Proof of Theorem I, (ii) *and* (iii).

2·1. It is convenient to work in terms of the series.

$$(2\cdot 11) \qquad (\mathrm{U})\ \Sigma\ \frac{(-1)^n}{n\sin n\theta\pi},\ (\mathrm{V})\ \Sigma\ \frac{(-1)^n}{\sin n\theta\pi}$$

[1] Here and later $A=A(\theta)$ denotes a positive number depending only on θ, whose precise value is immaterial.

(which differ only trivially from S and T, becoming S and T when θ is replaced by $\theta+1$). We shall also have to consider the series

$$(2\cdot12)\qquad (\mathrm{W})\ \sum \frac{1}{\sin^2 n\theta\pi}.$$

Lemma 1. *If ω and ω' are positive, and $\theta=\omega/\omega'$ belongs to Θ, or again if θ is algebraic, then*

$$(2\cdot13)\qquad f(s)=f(s,\theta)=\sum\frac{(-1)^n}{n^s\sin n\theta\pi}$$

is absolutely convergent when the real part σ of $s=\sigma+it$ is sufficiently large; and

$$(2\cdot14)\qquad \omega^{s-1}f(s,\theta)+\omega'^{\,s-1}f\left(s,\frac{1}{\theta}\right)$$

$$=\frac{(2\pi)^s}{\Gamma(s)\sin\frac{1}{2}s\pi}\,\zeta_2\left(1-s,\frac{\omega+\omega'}{2},\ \omega,\ \omega'\right),$$

where ζ_2 is the double zeta-function of Barnes.[1]

For the proof, see Hardy and Littlewood, 3 (Lemma β, p. 29). There we consider the algebraic case. If θ belongs to Θ (when of course it is not generally algebraic) the result holds for $\sigma>1$. See Hardy and Littlewood, 4 (Lemma 3, p. 216).

Suppose now in particular that

$$(2\cdot15)\qquad \theta=\frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\cdots=\sqrt{(a^2+1)}-a,$$

where a is odd. We take

$$\omega^2=\sqrt{(a^2+1)}-a,\qquad \omega'^2=\sqrt{(a^2+1)}+a.$$

Then $\quad f(s,\theta)=f(s,1/\theta)$ and

$$(2\cdot16)\qquad f(s)=\frac{(2\pi)^s\zeta_2(1-s,\frac{1}{2}\omega+\frac{1}{2}\omega',\ \omega,\ \omega')}{2\Gamma(s)\sin\frac{1}{2}s\pi\cosh\{(s-1)\log\omega'\}}$$

[1] Barnes, 7.

The zeta-function is regular all over the plane, except for simple poles at $s=0$ and $s=-1$, and vanishes when $s=2, 4, 6, \ldots\ldots\ldots$[1] Hence $f(s)$ is meromorphic, with poles at $s=0$, $s=-1$, and

$$s=1+(l+\tfrac{1}{2})\frac{\pi i}{\log \omega},$$

where l is an integer, positive or negative. If now we write

$$g(s)=g(s, \theta)=f(s+1, \theta+1)=\sum \frac{1}{n^{s+1} \sin n\theta\pi}, \tag{2·17}$$

we obtain

Lemma 2. *If $\theta=\sqrt{(a^2+1)}$, where a is an odd integer, then $g(s)$ is a meromorphic function of s, regular except for poles at $s=-1$, $s=-2$, and*

$$s=\frac{(2l+1)\pi i}{\log\left(\sqrt{(a^2+1)}-a\right)} \tag{2·18}$$

We may remark in passing that the main result of the lemma, that $g(s)$ is meromorphic, holds for all quadratic θ. The proof is in principle the same as that which we have given in the special case, but it is naturally more elaborate and (except when, as here, all the partial quotients of the continued fraction for θ are even) it is necessary to treat the two functions

$$\sum \frac{(-1)^n}{n^s \sin n\theta\pi}, \quad \sum \frac{1}{n^s \sin n\theta\pi}$$

simultaneously.[2]

2·2. The Rieszian mean of $\sum u_n$, of logarithmic type and order k, is[3]

$$U^k(w)=w^{-k} \sum_{\log n \leq w} u_n (w-\log n)^k \tag{2·21}$$

[1] See Barnes, pp. 338, 340. That the zeta-function which occurs here vanishes for $s=2, 4, \ldots\ldots$ follows from the regularity of $f(s)$ at those points, and may be verified directly from Barnes' contour integral.

[2] Compare Cooper's discussion of the function $\sum e^{n^2\theta\pi i}\, n^{-s}$ (Cooper, 10).

[3] Hardy and Riesz, 11, p. 21.

When $k=0$ and e^w is an integer, $U^k(w)$ reduces to U_n. If we suppose that, as here, the series $g(s)=\sum u_n n^{-s}$ is absolutely convergent for $\sigma>0$, we have [1]

$$(2{\cdot}22)\qquad U^k(w)=\frac{\Gamma(1+k)}{2\pi i\ w^k}\int_{c-i\infty}^{c+i\infty}\frac{g(s)}{s^{1+k}}e^{ws}\,ds.$$

The zeta-function in (2·16), and so $g(s)$, is of finite order (in the sense of Lindelöf and Bohr)[2] in any strip $\sigma_1\leqq\sigma\leqq\sigma_2$.[3] It follows that, if k is a sufficiently large integer, we may deform the path of integration into the line $(-b-i\infty,\ -b+i\infty)$, where $0<b<1$, if we introduce the appropriate corrections for the residues. We thus obtain

$$(2{\cdot}23)\qquad U^k(w)=\frac{\Gamma(1+k)}{2\pi i\, w^k}\int_{-b-i\infty}^{-b+i\infty}\frac{g(s)}{s^{1+k}}e^{ws}\,ds+\frac{\Gamma(1+k)}{w^k}\left(R^*+\sum R_l\right),$$

where R^* is the residue of the integrand at the origin and R_l a typical residue at a pole (2·18).

The residue R^* is the coefficient of s^k in $g(s)\,e^{ws}$, which is a polynomial of degree k in w whose leading term is

$$\frac{w^k}{k!}\,g(0).$$

The series $\sum R_l$ is

$$(2{\cdot}24)\qquad \sum\left\{\left(l+\tfrac{1}{2}\right)\frac{\pi i}{\log\omega}\right\}^{-1-k}\exp\left\{\left(l+\tfrac{1}{2}\right)\frac{w\pi i}{\log\omega}\right\}G_l,$$

where G_l is the residue of $g(s)$. Since $g(s)$ is of finite order, this series converges, absolutely, and uniformly in w, when k is sufficiently large, and is a periodic function of w, with period $4\log\omega$. Finally the integral in (2·23) is, for the same reason, the product of w^{-k} by an absolutely and uniformly convergent integral. It follows that

$$U^k(w)=g(0)+O(w^{-1})+O(w^{-k})=g(0)+o(1).$$

so that U is summable, to sum $g(0)$, by Riesz's means of sufficiently high order. So far our argument demands no unproved assumption. If we now assume the

[1] Hardy and Riesz, 11, p. 50.
[2] Hardy and Riesz, 11, p. 14.
[3] Hardy and Littlewood, 3, p. 31.

truth of clause (i) of Theorem 1, it follows, by the 'convexity theorem' for Rieszian means,[1] that U is summable, to sum $g(0)$, by means of any positive order.

2·3. We have thus (subject to our provisional assumption) proved clause (iii) of the theorem. To prove clause (ii) we consider Riesz's 'arithmetic' means, known to be equivalent to Cesàro's.[2] The arithmetic mean of U, of order k, is[3]

$$(2\cdot31)\quad U^{(k)}(w)=w^{-k}\sum_{n\leq w} u_n(w-n)^k,$$

and[4]

$$(2\cdot32)\quad U^{(k)}(w)=\frac{\Gamma(1+k)}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\,\frac{\Gamma(s)}{\Gamma(s+1+k)}\,w^s\,ds,$$

if $w>0$, $c>0$. The characteristic difference between this formula and (2·22) lies in the absence of the factor w^{-k} on the righthand side.

It is plain that we may transform (2·32) as we transformed (2·22). But there will be no factor w^{-k} multiplying the series which corresponds to the series (2·24) and which is, like that series, periodic in w. It will follow that $U^{(k)}(w)$ oscillates finitely for sufficiently large k, when $w\to\infty$, and this will prove (ii).

It is important to observe that we have proved incidentally, and without assuming the truth of (i), that the arithmetic means (and therefore the Cesàro means) of sufficiently high order are bounded. As this observation plays an essential part in the proof of (i), we state it formally in a lemma.

Lemma 3. *The Cesàro means of U, of sufficiently high order, are bounded.*

Proof of Theorem I (i).

3·1. We pass to the proof of (i). It now becomes necessary to take account of the properties of the continued fraction for θ.

[1] See Riesz, 15. Particular cases of the theorem had been proved before by ourselves and other writers.

[2] The only complete proof of the equivalence is that given (after Riesz) by Hobson, 13, pp. 90-98.

[3] Hardy and Riesz, 11, p. 23.

[4] Hardy and Riesz, 11, p. 51.

Throughout what follows O and o refer to the limit process $n \longrightarrow \infty$ or $s \longrightarrow \infty$;[1] the constants implied by the O's depend on θ (*i.e.* on a) only. Also $A = A(\theta)$ denotes generally a positive number depending only on θ, $\zeta = \zeta(\theta)$ a number between 0 and 1 depending only on θ.

We suppose that θ is defined by (2·15), and that

$$C_0 = \frac{p_0}{q_0} = \frac{0}{1}, \ C_1 = \frac{p_1}{q_1} = \frac{1}{2a}, \ \ldots C_s = \frac{p_s}{q_s}, \ \ldots$$

are the convergents to θ. Then $p_s = q_{s-1}$, and p_s and q_s are of opposite parity. We denote by $2a'$ the complete quotient corresponding to the partial quotient $2a$, so that

$$2a' = 2a + \theta = \sqrt{(a^2+1)} + a,$$

and write

$$q'_{s+1} = 2a' q_s + q_{s-1}.$$

Then

(3·11) $$q_{s+1} < A q_s, \ q_{s+1} > (1+A) q_s, \ q'_{s+1} > (1+A) q_{s+1}.$$

It is familiar that

(3·12) $$C_{s+1} - C_s = \frac{(-1)^s}{q_s q_{s+1}}, \ \theta - C_s = \frac{(-1)^s}{q_s q'_{s+1}}$$

If ν is an integer less than q_s, we have[2]

(3·13) $$|\nu\theta - i| \geqq |q_{s-1}\theta - p_{s-1}| = \frac{1}{q'_s}$$

for all integers i. Also

(3·14) $$\operatorname{cosec} \nu\theta\pi = O(\nu)$$

for all ν, and

(3·15) $$|\operatorname{cosec} \nu\theta\pi| > A\nu$$

for an infinity of ν (*e.g.* $\nu = q_s$).

3·2. We shall make repeated use of the following lemma, the form of which was suggested to us by Mr. E. C. Titchmarsh.

[1] s is a positive integer; the complex s of §2 does not appear again.

[2] Perron, **14**, p. 52.

Lemma 4. *Suppose that $s>1$, that $\nu<q_{s+1}$, and that ν is not a multiple of q_s. Then there is a positive $\zeta(\theta)$ less than 1 such that*

$$(3\cdot21)\qquad 1-\zeta<\left|\frac{\sin\nu\theta\pi}{\sin\nu C_s\pi}\right|<1+\zeta$$

for all ν for which either $\nu\theta$ or νC_s differs from an integer by less then $\frac{1}{4}$. And for all ν

$$(3\cdot22)\qquad \operatorname{cosec}\nu\theta\pi=O(|\operatorname{cosec}\nu C_s\pi|).$$

Write $\nu=rq_s+\mu$, where $r=[\nu/q_s]$, so that $0<\mu<q_s$ if $r<2a$ and $0<\mu<q_{s-1}$ if $r=2a$; and write

$$\nu C_s=\xi_\mu+f_\mu,\ \nu\theta=\xi'_\mu+f'_\mu,$$

where ξ_μ and ξ'_μ are integers and $|f_\mu|$ and $|f'_\mu|$ do not exceed $\frac{1}{2}$.[1] Then

$$f_\mu-f'_\mu=\xi'_\mu-\xi_\mu+\frac{(-1)^{s+1}\nu}{q_s q'_{s+1}}.$$

The lefthand side is *ex hypothesi* numerically less than $\frac{3}{4}$ and the last term on the righthand side is numerically less than $1/q_s<\frac{1}{4}$. It follows that $\xi'_\mu=\xi_\mu$ and that

$$|f_\mu-f_\mu'|<\frac{q_{s+1}}{q_s q'_{s+1}}<\frac{1}{(1+A)q_s}.$$

But $|f_\mu|\geq 1/q_s$. Hence f_μ and f'_μ have the same sign and

$$1-\zeta<f_\mu'/f_\mu<1+\zeta,$$

a result plainly equivalent to (3·21). As regards (3·22), this follows from (3·21) if either f_μ or f'_μ is less than $\frac{1}{4}$, and is trivial if neither is less than $\frac{1}{4}$.

[1] $|f_\mu|$ may be $\frac{1}{2}$, in which case there is ambiguity; we may agree then to take f_μ positive.

3·3. **Lemma 5.** $\sum_{1}^{q_s-1} \frac{1}{\sin^2 \nu C_s \pi} = O(q_s^2)$

When ν varies from 1 to q_s-1, f_μ assumes, each once, the values

$$\pm\lambda/q_s \; (\lambda=1, 2, \ldots\ldots, \lambda \leq \tfrac{1}{2} q_s)$$

the value $\frac{1}{2}q_s$ occurring, if at all, with one sign only. Hence

$$\Sigma \operatorname{cosec}^2 \nu C_s\pi = O\left\{q_s^2 \left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)\right\} = O(q_s^2),$$

which proves the lemma.

Lemma 6 [1]: $W_n(\theta) = \sum_{1}^{n} \frac{1}{\sin^2 \nu\theta\pi} = O(n^2)$.

Since q_{s+1}/q_s is bounded, it is enough to prove this when $n=q_s$. The term $n=q_s$ contributes $O(q_s^2)$. The remainder, by Lemmas 4 and 5, contribute

$$O(\Sigma \operatorname{cosec}^2 \nu C_s \pi) = O(q_s^2)$$

3·4. **Lemma 7**. *If p and q are coprime integers of opposite parity, and*

(3·41) $$B(p, q) = \sum_{\nu=1}^{q-1} (-1)^\nu \operatorname{cosec} \frac{\nu p \pi}{q},$$

then

(3·42) $$\frac{1}{q} B(p, q) + \frac{1}{p} B(q, p) = \frac{2}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{x^{p+q-1}}{(x^{2p}-1)(x^{2q}-1)} - \frac{1}{4pq} \frac{1}{(x-1)^2} - \frac{1}{4pq} \frac{1}{(x+1)^2} \right\} dx.$$

It is easily verified that the integrand is bounded for $x=1$ and $x=-1$. It follows from Cauchy's Theorem that the value of the integral is $4i$ times the sum of the residues of the integrand at poles above the real axis. Calculating these residues, we obtain (3·42).

Lemma 8. *If $C_s = p_s/q_s$ is a convergent to θ, then*

(3·43) $$B(q_s) = B(p_s, q_s) = O(q_s).$$

[1] This lemma is not actually used, and we include it because it is interesting in itself.

We denote the integral in (3·42) by $J(p, q)$. Making some obvious elementary transformations we obtain

$$J(p,q)=\frac{1}{2q}\int_0^\infty\left\{\frac{1}{\sinh w\sinh\phi w}-\frac{1}{4pq}\left[\frac{1}{\cosh^2(w/2q)}+\frac{1}{\sinh^2(w/2q)}\right]\right\}dw,$$

$$=\frac{1}{2q}\int_0^\infty\left\{\left[\frac{1}{\sinh w\sinh\phi w}-\frac{1}{\phi w^2}\right]-\frac{1}{4pq}\left[\frac{1}{\cosh^2(w/2q)}+\frac{1}{\sinh^2(w/2q)}-\left(\frac{2q}{w}\right)^2\right]\right\}dw,$$

where $\phi=p/q$. The integral here is

$$\int_0^\infty\left(\frac{1}{\sinh w\sinh\phi w}-\frac{1}{\phi w^2}\right)dw-\frac{1}{2\phi q}\int_0^\infty\frac{dw}{\cosh^2 w}$$

$$+\frac{1}{2\phi q}\int_0^\infty\left(1-\frac{w^2}{\sinh^2 w}\right)\frac{dw}{w^2}=\int_0^\infty\left(\frac{1}{\sinh w\sinh\theta w}-\frac{1}{\theta w^2}\right)dw+o(1)=O(1)$$

when $q\rightarrow\infty$, $p/q\rightarrow\theta$. It therefore follows from (3·42) that

(3·44) $$\frac{1}{q_s}B(p_s,q_s)+\frac{1}{p_s}B(q_s,p_s)=O\left(\frac{1}{q_s}\right)$$

when $s\rightarrow\infty$.

Now

$$B(q_s,p_s)=B(2aq_{s-1}+q_{s-2},q_{s-1})=B(q_{s-2},q_{s-1})=B(p_{s-1},q_{s-1}),$$

and so, from (3·44),

$$B(p_s,q_s)=O(1)+O\left(\frac{q_s}{q_{s-1}}\right)B(p_{s-1},q_{s-1})$$

$$=O(1)+O\left(\frac{q_s}{q_{s-1}}\right)+O\left(\frac{q_s}{q_{s-2}}\right)+\dots\dots,$$

on repeating the argument. Since q_s increases more rapidly than a geometrical progression, this is $O(q_s)$.

We may observe that the results of Lemmas 6 and 8 are in fact true for any θ of Θ and in particular any quadratic θ. Only slight changes are required in the proof of Lemma 6, or in that of Lemma 8 when all the partial quotients of θ are even. But in the general case the proof of Lemma 8 becomes more complicated because (as in the extension of Lemma 2, referred to at the end of § 2·1) it is then necessary to consider simultaneously sums of two slightly different types.

3·5. **Lemma 9.** $V_{q_s}(\theta)=\sum_{1}^{q_s}\frac{(-1)^{\nu}}{\sin\nu\theta\pi}=O(q_s)$.

The contribution of the term $\nu=q_s$ is $O(q_s)$. If $0<\nu<q_s$, we have

$$\sin\nu\theta\pi-\sin\nu C_s\pi=O\left(\left|\sin\tfrac{1}{2}\nu\pi(\theta-C_s)\right|\right)=O\left(\frac{1}{q_s}\right),$$

so that

$$V_{q_s}(\theta)-B(p_s,q_s)=O(q_s)+O\left\{\frac{1}{q_s}\sum_{1}^{q_s-1}\frac{1}{|\sin\nu\theta\pi\sin\nu C_s\pi|}\right\}$$

$$=O(q_s)+O\left(\frac{1}{q_s}\sum_{1}^{q_s-1}\frac{1}{\sin^2\nu C_s\pi}\right)=O(q_s),$$

by Lemmas 4 and 5. The result now follows from Lemma 8.

Lemma 10. $V_n(\theta)=\sum_{1}^{n}\frac{(-1)^{\nu}}{\sin\nu\theta\pi}=O(n)$.

This is the principal lemma. It has been proved (Lemma 9) when $n=q_s$. If $q_s<n<q_{s+1}$, we can write

$$n=bq_s+n_1,$$

where $1\leq b\leq 2a$ and $0\leq n_1<q_s$ if $b<2a$, $0\leq n_1<q_{s-1}$ if $b=2a$ (and also $n_1>0$ if $b=1$). It is enough to prove

$$V_n=O(q_s)\pm V_{n_1}, \tag{3·51}$$

since then, repeating the argument, we obtain

$$V_n=O(q_s)+O(q_{s-1})+\ldots=O(q_s)=O(n).$$

We write

$$(3\cdot 52)\qquad V_s = \sum_{r=0}^{b-1} \sum_{rq_s+1}^{(r+1)q_s} + \sum_{bq_s+1}^{bq_s+n_1} = \sum_{r=0}^{b-1} V_{s,r} + V'_{n_1},$$

say. We first consider $V_{s,r}$. We can omit the term $\nu=(r+1)q_s$ with error $O(q_s)$: in the remaining terms $\nu=rq_s+\mu$, where $0<\mu<q_s$. For such ν we have

$$\operatorname{cosec} \nu\theta\pi - \operatorname{cosec} \nu C_s\pi = O\left(\left|\sin\frac{\nu\pi}{2q_s q'_{s+1}} . \operatorname{cosec} \nu\theta\pi . \operatorname{cosec} \nu C_s\pi\right|\right)$$

$$= O\left(\frac{1}{q_s}\operatorname{cosec}^2 \nu C_s\pi\right),$$

by Lemma 4. Hence

$$(3\cdot 53)\quad V_{s,r} = O(q_s) + O\left(\frac{1}{q_s}\sum_{\mu=1}^{q_s-1}\frac{1}{\sin^2\mu C_s\pi}\right)$$

$$+(-1)^{r(q_s+p_s)}\sum_{\mu=1}^{q_s-1}\frac{-(1)^{\mu}}{\sin\mu C_s\pi} = O(q_s),$$

by Lemmas 5 and 8. It follows from $(3\cdot 52)$ and $(3\cdot 53)$ that

$$(3\cdot 54)\qquad V_s = O(q_s) + V'_{n_1}.$$

In the last term on the right we have $\nu=bq_s+\mu$, where $0<\mu\leqq n_1$, and

$$\nu\theta = (bq_s+\mu)\theta = bp_s + \mu C_s + O\left(\frac{1}{q_s}\right) = bp_s + \mu\theta + O\left(\frac{1}{q_s}\right),$$

$$\sin\nu\theta\pi - (-1)^{bp_s}\sin\mu\theta\pi = O\left(\frac{1}{q_s}\right),$$

$$\operatorname{cosec}\nu\theta\pi - (-1)^{bp_s}\operatorname{cosec}\mu\theta\pi = O\left(\frac{1}{q_s}\left|\operatorname{cosec}\nu\theta\pi\right|\left|\operatorname{cosec}\mu\theta\pi\right|\right)$$

$$= O\left(\frac{1}{q_s}\operatorname{cosec}^2\mu C_s\pi\right),$$

by Lemma 4. Hence

$$(3\cdot 55)\ V'_{n_1} = O\left(\frac{1}{q_\nu}\sum_{\mu=1}^{q_\nu-1}\frac{1}{\sin^2\mu\theta_\nu\pi}\right)+(-1)^{b(p_\nu+q_\nu)}\sum_{\mu=1}^{n_1}\frac{(-1)^\mu}{\sin\mu\theta\pi}$$

$$=O(q_\nu)\pm V_{n_1},$$

by Lemma 5. From (3·54) and (3·55) we deduce (3·51) and so the lemma.

3·6. **Lemma 11.** *If some Cesàro mean of* U *is bounded, and*

$$u_1+2u_2+\ldots+nu_n=O(n),$$

then U_n *is bounded.*

We write for convenience $u_0=0$. Then

$$U_n=\frac{U_0+U_1+\ldots+U_n}{n+1}+\frac{u_1+2u_2+\ldots nu_n}{n+1}.$$

Hence the difference between the Cesàro means of U, of orders 0 and 1, is bounded. If follows that the difference between the means of orders k and $k+1$ is bounded, and this proves the Lemma.

3·7. We can now complete the proof of the theorem. By Lemma 3, the Cesàro means of U, of sufficiently high order, are bounded. By Lemma 10, the second hypothesis of Lemma 11 is satisfied. It follows from Lemma 11 that U oscillates finitely.

Further Results.

4·1. We add a few remarks about the behaviour of our series for general quadratic θ or general θ of class Θ, without attempting to justify all that we say in detail.

The arguments of §§ 3·2—3·5 are extensible, without new difficulties of principle, to any θ of Θ; the results

$$(4\cdot 11)\qquad V_n=O(n),\qquad W_n=O(n^2)$$

of Lemmas 10 and 6 hold for all such θ. When the quotients of θ are all even, but little elaboration is needed; in the general case there is the complication alluded to in § 3·4. The results (4·11) are obviously 'best possible' for any θ.

The situation in regard to U is a little more complex. It is easy to prove that

$$(4{\cdot}12)\qquad U_n = O(\log n)$$

for all θ of Θ. For some θ, as we have seen, more is true (U_n being bounded), but (4·12) is the most that is true even for all *quadratic* θ. For quadratic θ, in fact, there are only two possibilities, *viz.* $U_n = O(1)$ and

$$(4{\cdot}13)\qquad U_n = A \log n + O(1) \qquad (A \neq 0).$$

It is interesting to give an example of the second case.

Recurring to the analysis of §2·1, let

$$\theta = \frac{1}{2a+}\frac{1}{2b+}\frac{1}{2a+}\frac{1}{2b+}\cdots,$$

where $a \neq b$. Then

$$\theta = \frac{1}{2a+\theta_1}, \quad \theta_1 = \frac{1}{2b+\theta}, \ldots$$

where

$$\theta = \sqrt{\left(\frac{b}{a}\right)}\{\sqrt{(ab+1)} - \sqrt{(ab)}\}, \quad \theta_1 = \sqrt{\left(\frac{a}{b}\right)}\{\sqrt{(ab+1)} - \sqrt{(ab)}\}.$$

If

$$h(s) = \sum \frac{(-1)^n}{n^{s+1} \sin n\theta\pi},$$

we obtain, by analysis similar to that of §2,

$$h(s) = \frac{(2\pi)^{s+1}}{\Gamma(1+s)\cos\frac{1}{2}s\pi}\,\frac{1}{1-(\theta\theta_1)^s}\,\left\{\zeta_2\left(-s, \frac{1+\theta}{2}, 1, \theta\right) - \theta^s \zeta_2\left(-s, \frac{1+\theta_1}{2}, 1, \theta_1\right)\right\}.$$

The arithmetic mean of U of order k is given by (2·32), with $h(s)$ in place of $g(s)$. The difference is that there is now a double pole at the origin. Otherwise we may argue as in § 2·3, and we find that $U^{(k)}(w)$ is, for sufficiently large k, of the form

$$A \log w + O(1),$$

where

$$A = -\frac{(b-a)\pi}{12 \log\{\sqrt{(ab+1)} - \sqrt{(ab)}\}}.$$

It follows (substantially, as in the proof of Lemma 11) that U_n itself is of the same form. If in particular $a=2$, $b=1$, we find

$$\theta=\tfrac{1}{2}\sqrt{6}-1,\ A=\frac{\pi}{12\log(\sqrt{3}-\sqrt{2})},$$

$$\sum_{1}^{n}\frac{1}{\nu\sin\frac{1}{2}\nu\pi\sqrt{6}}=\frac{\pi\log n}{12\log(\sqrt{3}-\sqrt{2})}+O(1).$$

We add in conclusion that

$$\sum_{1}^{n}\frac{1}{|\sin\nu\theta\pi|}=O(n\log n),\quad \sum_{1}^{n}\frac{1}{\nu|\sin\nu\theta\pi|}=O((\log n)^{2})$$

for all θ of Θ, and that these results also are the best possible of their kind.[1]

Bibliographical Note.

Our own relevant writings have been published in various journals under the general title 'Some Problems of Diophantine Approximation.' We may refer in particular to

1. Some Problems of Diophantine Approximation: *Proc. Fifth International Congress* (1912), 223-229 (a preliminary sketch).

2. The Trigonometrical Series associated with the Elliptic Theta-functions: *Acta. Math.*, 37 (1914), 193-238 [an additional note in *Proc. Camb. Phil. Soc.*, 21 (1923), 1-5].

3. The Lattice Points of a Right-angled Triangle: *Proc. London Math. Soc.* (2), 20 (1921), 15-36.

4 The Lattice Points of a Right-angled Triangle (second memoir): *Hamburg. Math. Abhandlungen*, 1 (1922), 212-249.

5. The Analytic Character of the Sum of a Dirichlet's Series considered by Hecke, *ibid*, 3 (1923), 57-68.

6. The Analytic Properties of Certain Dirichlet's Series associated with the Distribution of Numbers to Modulus Unity: *Trans. Camb. Phil. Soc.*, 22 (1923), 519-533.

[1] See Hardy and Littlewood, 4, p. 216.

The other books and memoirs referred to in the text are—

7. E. W. Barnes: A Memoir on the Double Gamma-function: *Phil. Trans. Royal Soc.* (A), 196 (1901), 265-387.

8. H. Behnke: Über die Verteilung von Irrationalitäten mod. 1: *Hamburg Math. Abhandlungen*, 1 (1922), 252-267.

9. H. Behnke: Zur Theorie der diophantischen Approximationen: *ibid*, 3 (1924), 261-318 and 4 (1926), 33-46.

10. R. Cooper: The Behaviour of Certain Series associated with limiting Cases of Elliptic Theta-functions: *Proc. London Math. Soc.*, 27 (1928), 410-426.

11. G. H. Hardy and M. Riesz: The General Theory of Dirichlet's Series: *Camb. Math. Tracts*, 18 (1915).

12. E. Hecke: Über analytische Functionen und die Verteilung von Zahlen mod. Eins: *Hamburg Math. Abhandlungen*, 1 (1922), 54-76.

13. E. W. Hobson: The Theory of Functions of a Real Variable, Vol. 2, second edition, 1926.

14. O. Perron: Die Lehre von den Kettenbrüchen, 1913.

15. M. Riesz: Sur un théorème de la moyenne et ses applications: *Acta Univ. Hungaricae*, 1 (1923), 3-15

Further references to the older literature connected with series of the type considered here will be found in a paper by Hardy, 'On Certain Series of Discontinuous Functions connected with the Modular Functions,' *Quarterly Journal*, 36 (1905), 93-123.

ON MAHĀVĪRA'S SOLUTION OF RATIONAL TRIANGLES AND QUADRILATERALS

By

BIBHUTIBHUSAN DATTA

(University of Calcutta)

The subject of the solution of rational triangles and quadilaterals was a very favourite one with the early Hindu mathematicians. For it is found to have been treated in almost all the Hindu treatises on mathematics coming after the sixth century of the Christian era, that are available now. Mahāvīra's investigations[1] in this particular field deserve special consideration for more reasons than one. Indeed they have certain notable features which we miss in the works of others. Firstly, unlike other Hindu mathematicians who have simply noted the solutions of rational triangles and quadrilaterals obtained by or known to them, Mahāvīra has in most cases explicitly stated his methods of finding the solutions. In some cases, he is found to have proceeded no further than a mere indication of the methods of a solution. Secondly, he has treated certain problems of rational triangles and quadrilaterals which are not found elsewhere. Thirdly, in the treating of other common problems he has sometimes introduced modifications and improvements upon the works of his predecessors. What is more important for the general History of Mathematics, certain methods of finding solutions of rational triangles, the credit for the discovery of which should very rightly go to Mahāvīra, are attributed by modern historians, by mistake, to writers posterior to him. Indeed, no notice has been taken of Mahāvīra's investigations in the monumental history of Professor L. E. Dickson on the theory of numbers.[2] Absence of a comprehensive critical exposition in modern algebraic symbols and notations and a study from historical stand-point is

[1] *Gaṇita-sāra-saṁgraha* of Mahāvīra was edited with English translation and notes by M. Rangācārya in 1912, Madras.

[2] L. E. Dickson, *History of the Theory of Numbers*, 3 vols., Washington, 1919-1923; hereafter referred to as Dickson, *Numbers*.

perhaps responsible for it.[1] So it is highly desirable that an attempt should be made to secure a proper recognition for the works of Mahāvīra in the theory of numbers.

Definitions.

In order to understand correctly the real significance of the rules of Mahāvīra relating to his solutions of rational triangles and quadrilaterals we must be familiar with his mode of expression and definition of a few technical terms. A triangle or a quadrilateral whose sides, altitudes and other dimensions can be expressed in terms of *rational* numbers, Māhāvīra calls *janya* meaning "generated," "formed" or "that which is generated or formed."[2] These figures are of course always formed from particular numbers. Numbers which are necessary or are employed in forming a particular figure are called its *bīja-saṁkhyā* ("element number") or simply *bīja* ("element" or "seed"). Thus Mahāvīra says, "Forming, O friend! the generated figure from the *bīja* 3, 5,"[3] "forming another from half the base of the figure (rectangle) from the *bīja* 3, 5,"[4] etc. It is much noteworthy that Mahāvīra's mode of expression in this respect very closely resembles that of Diophantus who also says, "Form now a right-angled triangle from 7, 4," "forming a right-angled triangle from 8, 1."[5] It is strange that Mahāvīra never speaks of "right-angled triangle" though he has such terms as "isosceles" (*dvisama*) or "scalene" (*viṣama*) triangle.[6] What Diophantus or we call "forming a right-angled triangle from m, n," Mahāvīra calls "forming a longish quadrilateral or rectangle from m, n." In Hindu geometry, the term *karṇa* denotes the "hypotenuse" of a right-angled triangle as well as the "diagonal" of a rectangle (or of a quadrilateral). This clearly reveals the origin of the Hindu conception of a right-angled triangle.

[1] Much in this respect has been done before by Rangācārya who presented some of Mahāvīra's rules in modern algebraic symbols and notations. Information about Mahāvīra's contribution to mathematics in general will be found in an article by the present writer on "Hindu Contributions to Mathematics" in the *Bulletin of the Mathematical Association, University of Allahabad*, Vols. I and II; hereafter referred to as *Hindu Contributions*.

[2] *Gaṇita-sāra-saṁgraha*, vii. 90½ (introductory lines to). The section of Mahāvīra's work devoted to the treatment of rational triangles and quadrilaterals bears the sub-title "*Janya-vyavahāra* ("Janya operation") and it begins as "Hereafter we shall give out the *Janya* operation in calculations relating to measurement of areas."

[3] "बीजे द्वे त्रीणि सखे क्षेत्रे जन्ये तु संस्थाप्य," vii. 92½.

[4] "द्वि त्रिबीजक्षेत्र भुजार्धेन चान्यमुत्थाप्य," vii. 111/1/2½.

[5] *Arithmetica*, Book III, 19. *Cf.* T. L. Heath, *Diophantus of Alexandria*, 2nd edition, 1910, Cambridge, p. 167; hereafter this work will be referred to as Heath, *Diophantus*.

[6] Mahāvīra recognised only three kinds of triangles, equilateral, isosceles and scalene (*Gaṇita-sāra-saṁgraha*, vii. 4.)

Rational Right Triangles.

Mahāvīra has given the following rule for the solution of the equation,

$$x^2+y^2=z^2$$

in *rational integers*:

"For a rectangle which can be formed (from the *bīja*), the difference of the squares (of the two *bīja*) will be the perpendicular, twice the product will be the base and the sum of the squares the diagonal."[1]

If m, n be any two *rational integers*, then the perpendicular-side, base and hypotenuse of the right triangle formed from them, will be

$$m^2-n^2,\ 2mn,\ m^2+n^2$$

This solution was first discovered by Brahmagupta (628),[2] who explicitly laid down the condition, which is obviously necessary, that m must not be equal *to* n.

Ganeśa (1545) further generalised these solutions by pointing out that other solutions of the rational right triangles can be obtained by multiplying them by any rational integer.[3] Thus if l be a *rational integer*,

$$(m^2-n^2)l,\ \ 2mnl,\ \ (m^2+n^2)l \qquad \dots \quad (2)$$

will be more general solutions of $x^2+y^2=z^2$. And we now know it that all positive integral solutions of this equation are given without duplication by (2) when m and n are relatively prime integers, not both odd and $m > n > o$, while l is a positive integer.[4]

It should be remarked that the above solutions are of fundamental importance in the Hindu investigations of rational triangles and quadrilaterals. For every problem in this connexion has been treated and solved by the Hindu mathematicians entirely with the help of them.

It is to be always borne in mind that with Mahāvīra, "forming a rectangle from the *bīja* m, n" means taking a rectangle with the perpendicular-side, base and diagonal as m^2-n^2, $2mn$, m^2+n^2 respectively. Here again Mahāvīra closely resembles Diophantus. For with the latter also "forming a right-angled triangle from 7, 4" means taking a right-angled triangle with sides 7^2-4^2, $2.\ 7.4$, 7^2+4^2 or 33, 56, 65.[5]

[1] *Gaṇita-sāra-saṁgraha*, vii. 90½.

[2] *Brāhma-sphuṭa-siddhānta*, edited by Sudhākara Dvivedī, Benares, 1902, xii. 33.

[3] Bibhutibhusan Datta, *Hindu Contributions*.

[4] L. E. Dickson, "Rational triangles and quadrilaterals," *Amer. Math. Monthly*, Vol. 28, 1921, pp. 244-250.

[5] *Arithmetica*, III. 19.

Right Triangles having a given Leg.

All rational right triangles having a given leg (a), that is, rational solutions of

$$x^2+a^2=z^2,$$

says Mahāvīra,[1] will be given by

$$\frac{1}{2}\left(\frac{a^2}{n}-n\right),\quad \frac{1}{2}\left(\frac{a^2}{n}+n\right) \qquad \dots \quad (1)$$

where n is any rational number. These solutions were known before to Brahmagupta.[2] It will be easily recognised that they can be derived from the general solutions of the rational right triangles, *viz.*,

$$(m^2-n^2)l,\ 2mnl,\ (m^2+n^2)l \qquad \dots \quad (2)$$

by dividing them by $2nl$ and putting $m=a$.

Mahāvīra[3] has again said that rational right triangles having a given leg (a), will be obtained from the *bīja*

$$\frac{1}{2}\left(\frac{a}{p}-p\right),\ \frac{1}{2}\left(\frac{a}{p}+p\right) \quad \text{or} \quad \frac{a}{2q},\ q$$

where p, q are rational integers. The right triangles formed from the first set of these values will be

$$a,\ \frac{1}{2}\left(\frac{a^2}{p^2}-p^2\right),\ \frac{1}{2}\left(\frac{a^2}{p^2}+p^2\right) \qquad \dots \quad (3)$$

and those form the second set

$$\frac{a^2}{4q^2}-q^2,\ a,\ \frac{a^2}{4q^2}+q^2 \qquad \dots \quad (4)$$

These solutions are practically the same as (1). Another solution of more general character is

$$a,\ \frac{2mna}{m^2-n^2},\ \left(\frac{m^2+n^2}{m^2-n^2}\right)a$$

Though it has not been explicitiy noted by Mahāvīra, it follows at once, as will be presently shown, from one of the methods of solution usually adopted by him.

[1] *Gaṇita-sāra-saṁgraha*, vii. 97½.
[2] *Brahma-sphuṭa-siddhānta*, xii. 35.
[3] *Gaṇita-sāra-saṁgraha*, vii. 95½.

Right Triangles having a given Hypotenuse.

For finding all rational right triangles having a given hypotenuse (c), that is, for rational solutions of

$$x^2+y^2=c^2,$$

Mahāvīra has proposed two methods. The first is a very ordinary one and consists in assuming an arbitrary value (p) for one of the legs, so that the other becomes at once determinate.[1] In this case the solutions are

$$p, \ \sqrt{c^2-p^2}, \ c$$

This method has been re-indicated in a slightly different way by stating that the required solutions will be obtained from[2]

$$\sqrt{(c+p^2)/2} \text{ and } \sqrt{(c-p^2)/2}$$

and hence they are

$$p^2, \ \sqrt{(c^2-p^4)}, \ c$$

These solutions are defective in the sense that $\sqrt{(c^2-p^2)}$, or $\sqrt{(c^2-p^4)}$ might not be rational unless p is suitably chosen.

Mahāvīra's second method of solution is of greater importance. He says[3]:

"Each of the various figures (rectangles) that can be formed from the *bīja* are put down; by its diagonal is divided the given diagonal. The perpendicular, base and diagonal (of this figure) multiplied by this quotient (give rise to the corresponding sides of the figure having the given hypotenuse)."

Thus having obtained the general solutions of the rational right triangles, *viz.*,

$$m^2-n^2, \quad 2mn, \quad m^2+n^2$$

Mahāvīra reduces them in the ratio $c/(m^2+n^2)$, so that all rational right triangles having a given hypotenuse (c) will be

$$\left(\frac{m^2-n^2}{m^2+n^2}\right)c, \quad \frac{2mnc}{m^2+n^2}, \ c$$

[1] *Gaṇita-sāra-saṁgraha*, vii. 97½.

[2] *Ibid*, vii. 95½.

[3] *Ibid*, vii. 112½; compare also vii. 221½.

Bhāskara[1] states, without proof, the solutions

$$\frac{2nc}{n^2+1}, \quad \left(\frac{n^2-1}{n^2+1}\right)c, \; c$$

which are particular cases of Mahāvīra's solutions and have very probably been obtained by his method.

The second method of Mahāvīra is analogous to the Rule of Single False Position in Algebra. It plays an important rôle in his mathematics. Indeed it has been a powerful weapon at his hands in solving certain geometrical problems leading to indeterminate equations of the second degree. This method was later on re-discovered in Europe by Leonardo Fibonacci of Pisa (1202) and Vieta (c. 1580). Professor Dickson is not right in attributing the credit for its discovery to Leonardo.[2]

Problems involving Areas and Sides.

Mahāvīra proposes to find rational rectangles (or squares) in which the area will be *numerically* (*saṁkhyayā*) equal to any multiple or sub-multiple of the side, diagonal or perimeter or of linear combination of two or more of them.[3] Expressed symbolically, the problem is to solve

$$x^2+y^2=z^2 \qquad \dots (1)$$

$$rxy=mx+ny+pz \qquad \dots (2)$$

m, n, p, r being known rational numbers including, except in case of r, zero.

The method adopted for solution is the same as the second one in the previous case. Starting with any rational solution of

$$x'^2+y'^2=z'^2,$$

says Mahāvīra, calculate the value of

$$mx'+ny'+pz'=Q, \text{ say.}$$

[1] *Līlāvatī*; H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*, London, 1817, p. 61; hereafter referred to as Colebrooke, *Hindu Algebra*.

[2] Dickson, *Numbers*, II, p. 167.

[3] *Gaṇita-sāra-saṁgraha*, vii, 112½.

Then he observes that the required solutions of (1) and (2) will be obtained by reducing the values of x', y', z' in the ratio $Q/rx'y'$. Thus

$$\begin{aligned} x &= x'Q/rx'y' = Q/ry', \\ y &= y'Q/rx'y' = Q/rx', \\ z &= z'Q/rx'y'. \end{aligned} \qquad \dots \quad (3)$$

It can be readily verified that these values of x, y, z satisfy the equations (1) and (2).

In particular, let $m=n=0$, $p=1$, $r=\frac{1}{2}$; then we get

$$\frac{m^2+n^2}{mn}, \quad \frac{2(m^2+n^2)}{m^2-n^2}, \quad \frac{(m^2+n^2)^2}{mn(m^2-n^2)}$$

as the sides of a right triangle whose area equals the hypotenuse. These solutions were known to Bhāskara.[1]

Problems involving Sides but not Areas.

Mahāvīra obtained right triangles the sum of whose sides multiplied by arbitrary rational numbers has a given value.[2] Algebraically, the problems require the solution of

$$\left.\begin{aligned} x^2+y^2 &= z^2, \\ rx+sy+tz &= A, \end{aligned}\right\} \qquad \dots \quad (1)$$

where r, s, t, A are known rational numbers. Starting with the general solution of

$$x'^2+y'^2=z'^2$$

we are asked to calculate the value of

$$rx'+sy'+tz'=P, \text{ say.}$$

Then, says Mahāvīra, the solution of (1) will be

$$x=x'A/P, \; y=y'A/P, \; z=z'A/P.$$

[1] *Bījagaṇita*, edited by Sudhākara Dvivedī and Muralidhar Jha, Benares, 1927, p. 56.

[2] *Gaṇita-sāra-saṁgraha*, vii. 112½, 118½, 119½.

In particular, when $r=s=0$, $t=1$, $A=c$, then $P=z'$. So with the general solution m^2-n^2, $2mn$, m^2+n^2 of $x'^2+y'^2=z'^2$, we shall obtain the solution already given,

$$\left(\frac{m^2-n^2}{m^2+n^2}\right)c, \quad \frac{2mnc}{m^2+n^2}, \quad c$$

of rational right triangles having the hypotenuse c.

Similarly, putting $s=t=0$, $r=1$, $A=a$, we get

$$a, \quad \frac{2mna}{m^2-n^2}, \quad \left(\frac{m^2+n^2}{m^2-n^2}\right)a$$

as the solution of rational right triangles having the leg a. Solutions very nearly equal to this are noted by Bhāskara.[1]

If $r=s=2$, $t=0$, $A=1$, then $P=2(m^2-n^2+2mn)$. Hence all rectangles having the same perimeter unity will be

$$\frac{m^2-n^2}{2\,(m^2-n^2+2mn)}, \quad \frac{mn}{m^2-n^2+2mn}$$

where m, n are any rational numbers.[2] The isoperimetric right triangles will be given by

$$\left(\frac{m-n}{2m}\right)s, \quad \frac{ns}{m+n}, \quad \left\{\frac{m^2+n^2}{2m(m+n)}\right\}s$$

where s is the given perimeter.

Rational Isosceles Triangles.

Mahāvīra gives the following "rule for obtaining an isosceles triangle from a single generated rectangle."[3]

"In this isosceles triangle (required), the two diagonals (of the generated rectangle) are the two sides, twice the base (of the rectangle) is the base, the perpendicular-side is the altitude and the area (of the rectangle) is the area."

[1] *Līlāvatī*; Colebrooke, *Hindu Algebra*, p. 61.
[2] *Gaṇita-sāra-saṁgraha*, vii. 118½.
[3] *Gaṇita-sāra-saṁgraha*, vii. 108½.

Thus the sides of all rational isosceles triangles formed from m, n, are

$$m^2+n^2, \quad m^2+n^2, \quad 2(m^2-n^2)$$

$$\text{or} \quad m^2+n^2, \quad m^2+n^2, \quad 4mn$$

The altitude of the former is $2mn$ and of the latter m^2-n^2 and the area in either case is the same $2mn(m^2-n^2)$.

It should be noted that the device employed to find the above solutions is to juxtapose two rational right triangles (equal in this case) so as to have a common leg. The credit for the invention of this device is not due to Mahāvīra. It was adopted before him, though not explicitly stated, by Brahmagupta[1] to find rational triangles. It is indeed a very powerful device. For every rational triangle or quadrilateral may be formed by juxtaposing two or four rational right triangles, so that it suffices to know only the complete solution of $x^2+y^2=z^2$ in rational integers. Professor Dickson[2] has shown that with the help of this device it is possible to make a material simplification in Kummer's classic investigation of rational quadrilaterals.

Rational Scalene Triangle.

To find every scalene triangle having rational sides, area, altitude and segments of the base, Mahāvīra gives the rule :[3]

"Half the base of a derived (rectangle) is divided by any (optional) number. With this divisor and the quotient is obtained another rectangle. The sum of the perpendiculars (of these two rectangles) will be the base of the scalene triangle, the two diagonals its sides and the base (of the either rectangle) its altitudes."

If m, n, be any two rational numbers, the rational rectangle (AB′BH)

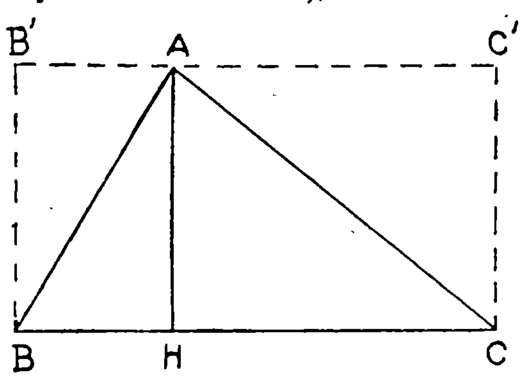


FIG. 1.

[1] *Brāhma-sphuṭa-siddhānta*, xii. 38 ff.

[2] *Amer. Math. Monthly*, Vol. 28, 1921, pp. 245-250.

[3] *Gaṇita-sāra-saṁgraha*, vii. $10\frac{1}{2}$.

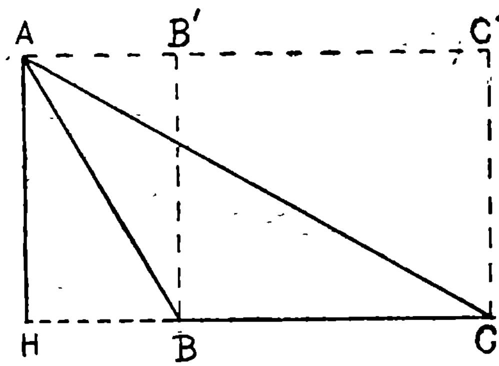


FIG. 2.

formed from them will be

$$m^2-n^2,\quad 2mn,\quad m^2+n^2.$$

If p, q be any two rational factors of mn, that is, if $mn=pq$, the second rectangle (AC′CH) will be obtained as

$$p^2-q^2,\quad 2pq,\quad p^2+q^2.$$

Juxtaposing these two rectangles so that they do not overlap (Fig. 1), the sides of the rational scalene triangle will be obtained as

$$m^2+n^2,\ p^2+q^2,\ (m^2-n^2)+(p^2-q^2)$$

where $mn=pq$. Evidently the two rectangles can be juxtaposed so as to overlap (Fig. 2). So the general solutions will be

$$m^2+n^2,\ p^2+q^2,\ (m^2-n^2)\pm(p^2-q^2).$$

The altitudes of the rational triangle obtained thus is $2mn$ or $2pq$, its area $mn(m^2-n^2)\pm pq(p^2-q^2)$ and the segments of its base are m^2-n^2 and p^2-q^2.

These solutions of the rational scalene triangles were restated in 1812 by J. Cunliffe.[1] The method of solution of a rational scalene triangle by juxtaposing two rational right triangles so as to have a common leg was discovered, as has been just stated, by Brahmagupta (628). In Europe, it is found to have been employed first by Bachet (1621).[2] But while Bachet

[1] Dickson, *Numbers* II, p. 193.

[2] *Ibid*, p. 192. Bachet's results may be briefly stated thus: Assuming any number, say 12, for the common leg, we are to find two squares such that the sum of each and 12^2 is a square: $35^2+12^2=37^2$, $16^2+12^2=20^2$. Hence by juxtaposition will be obtained the rational triangle with sides 37, 20, $35+16=51$ and altitude 12.

employed it to a very particular problem, Brahmagupta gave the general solutions[1]

$$\frac{1}{2}\left(\frac{m^2}{p}+p\right),\ \frac{1}{2}\left(\frac{m^2}{q}+q\right),\ \frac{1}{2}\left(\frac{m^2}{p}-p\right)+\frac{1}{2}\left(\frac{m^2}{q}-q\right)$$

where m, p, q are any rational numbers, as the sides of a rational scalene triangle whose altitude is m. So the credit for the discovery of this method of finding rational scalene triangles should rightly go to Brahmagupta or Mahāvīra, but not to Bachet as is supposed by Dickson.

Euler (*c.* 1750) noted that the sides of a rational scalene triangle will be proportional to

$$\frac{m^2+n^2}{mn},\ \frac{p^2+q^2}{pq},\ \frac{m^2-n^2}{mn}\pm\frac{p^2-q^2}{pq}=\frac{(mq\pm np)(mp\mp nq)}{mnpq}$$

and its altitude is proportional to 2. This result appears in a posthumous paper of Euler but the portion of that paper containing the proof is missing. It is probable that he employed Brahmagupta's method.

Rational Isosceles Trapezium.

Mahāvīra has shown how to obtain an isosceles trapezium whose sides, diagonals, altitude, segments and area can all be expressed in *rational* numbers. He says:[2]

"For an isosceles trapezium, the sum of the perpendicular of the first generated rectangle and the perpendicular of the second rectangle which is formed from any (*rational*) divisor of half the base of the first and the quotient will be the base; their difference will be the face; the smaller of the diagonals (of the generated rectangles) will be the flank side; the smaller perpendicular will be the segment; the greater diagonal will be the diagonal (of the isosceles trapezium); the greater area will be the area and the base (of the either rectangles) will be the altitude."

The first rectangle (AA′DH) generated from m, n, is

$$m^2-n^2,\quad 2mn,\quad m^2+n^2.$$

If p, q be any two rational factors of half the base of this ractangle, that is, if $pq=mn$, the second rectangle (AB′CH) from these factors will be

$$p^2-q^2,\quad 2pq,\quad p^2+q^2.$$

[1] *Brāhma-sphuṭa-siddhānta*, xii. 34.

[2] *Gaṇita-sāra-saṁgraha*, vii. 99½.

By judiciously juxtaposing these two rectangles, we shall obtain an isosceles trapezium of the type required (ABCD):

$$CD=(p^2-q^2)+(m^2-n^2),$$

$$AB=(p^2-q^2)-(m^2-n^2),$$

$$AD=BC=m^2+n^2, \quad \text{if } m^2+n^2<p^2+q^2$$

$$DH=m^2-n^2, \quad \text{if } m^2-n^2<p^2-q^2$$

$$AC=BD=p^2+q^2, \quad \text{if } p^2+q^2>m^2+n^2$$

$$AH=2mn=2pq,$$

$$\text{Area } ABCD=2pq(p^2-q^2), \quad \text{if } 2pq(p^2-q^2)>2mn(m^2-n^2)$$

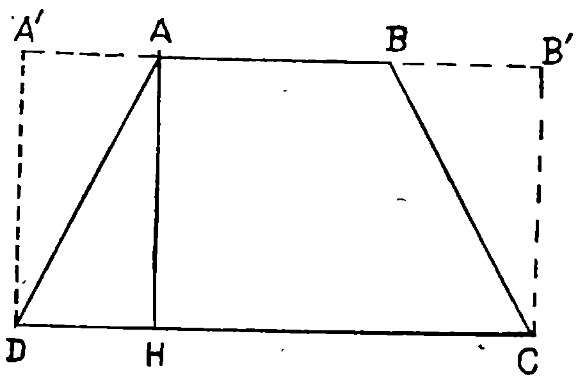


FIG. 3.

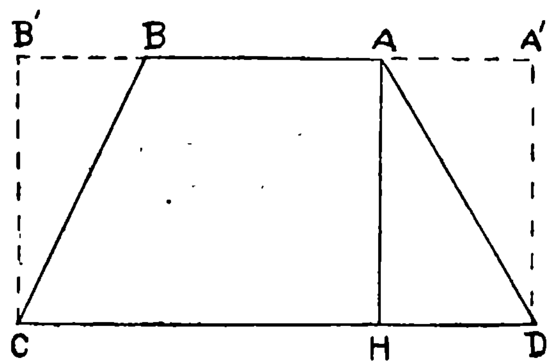


FIG. 4

The necessity of the conditions $m^2+n^2<p^2+q^2$, $m^2-n^2<p^2-q^2$, etc., will be at once realised from a glance at the Figs. 1 and 2. The above specifications of the dimensions of a rational isosceles trapezium will give the Fig. 3. But when the conditions are reversed $m^2+n^2>p^2+q^2$, $m^2-n^2>p^2-q^2$, $2pq(p^2-q^2)<2mn(m^2-n^2)$ (Fig. 4), the dimensions of the isosceles trapezium should be specified as

$$\text{CD}=(m^2-n^2)+(p^2-q^2),$$

$$\text{AB}=(m^2-n^2)-(p^2-q^2),$$

$$\text{AD}=\text{BC}=(p^2+q^2),$$

$$\text{DH}=p^2-q^2,$$

$$\text{AC}=\text{BD}=m^2+n^2,$$

$$\text{AH}=2mn=2pq,$$

$$\text{Area ABCD}=2mn(m^2-n^2).$$

The problem to find a rational isosceles trapezium was attempted earlier by Brahmagupta. Though the method employed is the same, *viz.*, the juxtaposition of suitable right triangles, his solutions are entirely different. He simply noted the results[1]

$$\text{CD}=\frac{1}{2}\left(\frac{4m^2n^2}{p}-p\right)+(m^2-n^2),$$

$$\text{AB}=\frac{1}{2}\left(\frac{4m^2n^2}{p}-p\right)-(m^2-n^2),$$

$$\text{AD}=\text{BC}=m^2+n^2,$$

without specifying the necessary conditions. So Mahāvīra's investigations of the problem are fuller than those of his predecessor.

Trapezium with Three Equal Sides.

This problem is very nearly the same as the one just discussed with this difference that in this case one of the parallel sides also is equal to the slanting sides. The method indicated for its solution is as follows : [2]

[1] *Brāhma-sphuṭa-siddhānta*, xii. 36.

[2] *Gaṇita-sāra-saṁgraha*, vii. 101½.

" For a trapezium with three equal sides, (proceed) as in the case of the isosceles trapezium, with (the rectangles formed from) the quotient of the area of a generated rectangle divided by the square root of its side multiplied by the difference of its *bīja* and the divisor, and from the side and perpendicular."

The rectangle from m, n is

$$m^2-n^2, \quad 2mn, \quad m^2+n^2$$

Calculate $$\frac{2mn\times(m^2-n^2)}{\sqrt{2mn}\times(m-n)}=\sqrt{2mn}\,(m+n)$$

Then from $\sqrt{2mn}\,(m-n)$, $\sqrt{2mn}\,(m+n)$ form the second rectangle

$$8m^2n^2, \quad 4mn(m^2-n^2), \quad 4mn(m^2+n^2)$$

Again from $2mn$, m^2-n^2 form the third rectangle

$$6m^2n^2-m^4-n^4, \quad 4mn(m^2-n^2), \quad (m^2+n^2)^2$$

By the juxtaposition of the second and third rectangles we get a rational integral trapezium ABCD whose sides AB, BC, AD are equal.

$$\mathrm{CD}=8m^2n^2+(6m^2n^2-m^4-n^4)=14m^2n^2-m^4-n^4,$$

$$\mathrm{AB}=8m^2n^2-(6m^2n^2-m^4-n^4)=(m^2+n^2)^2,$$

$$\mathrm{AD}=\mathrm{BC}=(m^2+n^2)^2, \quad \text{if} \quad m^2+n^2<4mn,$$

$$\mathrm{CH}=6m^2n^2-m^4-n^4,$$

$$\mathrm{AH}=4mn(m^2-n^2),$$

$$\mathrm{AC}=\mathrm{BC}=4mn(m^2+n^2),$$

$$\text{Area ABCD}=32m^3n^3(m^2-n^2).$$

This problem was attacked also by Brahmagupta who arrived at the same solutions by some method not indicated.[1]

Rational Quadrilateral.

Mahāvīra gives the following "rule for finding the sides, face, base, altitudes, diagonals, segments and the area of a quadrilateral of unequal sides" in terms of rational numbers:[2]

[1] *Brāhma-sphuṭa-siddhānta*, xii. 37.

[2] *Gaṇita-sāra-saṃgraha*, vii. 108½.

"The base and the perpendicular (of the smaller and larger derived rectangles of reference) being multiplied reciprocally by the longer and shorter diagonals and (each again) by the shorter diagonal will be the sides, the face and the base (of the required quadrilateral). The perpendiculars and bases are reciprocally multiplied and then added together ; again the product of the perpendiculars is added to the product of the bases. These two sums multiplied by the smaller diagonal will be the diagonals. (Those sums) when multiplied (respectively) by the base and perpendicular of the smaller figure (of reference) will be the altitudes and they when multiplied (respectively) by the perpendicular and the base will be the segments of the base. Other segments will be the difference of these and the base. Half the product of the diagonals (of the required figure) will be the area."

Thus if we have two rectangles

$$m^2-n^2,\ 2mn,\ m^2+n^2$$

$$p^2-q^2,\ 2pq,\ p^2+q^2$$

where m, n, p, q are rational integers and if the figure from m, n be smaller than the other figure, then, according to Mahāvīra, we shall obtain a rational quadrilateral of which the sides are

$$2mn(p^2+q^2)(m^2+n^2),\ (m^2-n^2)(p^2+q^2)(m^2+n^2),\ 2pq(m^2+n^2)^2,$$
$$(p^2-q^2)(m^2+n^2)^2$$

whose diagonals are

$$\{2pq(m^2-n^2)+2mn(p^2-q^2)\}(m^2+n^2),\ \{(m^2-n^2)(p^2-q^2)+4mnpq\}(m^2+n^2)$$

whose altitudes are

$$\{2pq(m^2-n^2)+2mn(p^2-q^2)\}2mn,\ \{(m^2-n^2)(p^2-q^2)+4mnpq\}(m^2-n^2)$$

whose segments are

$$\{2pq(m^2-n^2)+2mn(p^2-q^2)\}(m^2-n^2),$$
$$\{(m^2-n^2)(p^2-q^2)+4mnpq\}2mn$$

and whose area is

$$\tfrac{1}{2}\{2pq(m^2-n^2)+2mn(p^2-q^2)\}\{(m^2-n^2)(p^2-q^2)+4mnpq\}(m^2+n^2)^2$$

Quadrilaterals of this kind are called Brahmagupta quadrilaterals. It was Brahmagupta who first proposed for solution the remarkable proposition of finding the rational inscribed quadrilaterals and also succeeded in obtaining solutions of the same.[1] Though Mahāvīra has followed the method of Brahmagupta his solutions differ in certain respects from those of the latter. Bhāskara has improved these solutions much further.[2]

Figures with a given Area.

The sides of a *rectangle* having a given area A, says Mahāvīra,[3] will be p, q, where p, q are any two of rational factors of A, that is, $A=pq$

Sides of an *isosceles trapezium* having a given area A will be, says Mahāvīra,[4]

$$\text{top}=\frac{s^2A-2mn(m^2-n^2)}{2mns}$$

$$\text{base}=\frac{1}{s}\left\{\frac{s^2A-2mn(m^2-n^2)}{2mn}+2(m^2-n^2)\right\}$$

$$=\frac{s^2A+2mn(m^2-n^2)}{2mns}$$

$$\text{side}=\frac{m^2+n^2}{s}$$

$$\text{altitude}=\frac{2mn}{s}$$

where s is an arbitrary rational number chosen such that $s^2A>2mn(m^2-n^2)$.

Sides of a *trapezium with three equal sides* having a given area A, will be[5]

$$\text{side}=\tfrac{1}{2}\left(\frac{A^2}{s^3}+s\right)$$

$$\text{base}=2s-\tfrac{1}{2}\left(\frac{A^2}{s^3}+s\right)$$

$$\text{altitude}=\frac{A}{s}$$

where s is an arbitrary rational number. The *rationale* of this solution can be easily determined.

[1] *Brāhma-sphuṭa-siddhānta*, xii. 38.
[2] Bibhutibhusan Datta, *Hindu Contributions.*
[3] *Gaṇita-sāra-saṁgraha*, vii. 146.
[4] *Ibid*, vii. 148.
[5] *Ibid*, vii. 150.

Similarly Mahāvīra considered triangles and quadrilaterals of unequal sides, having a given area.[1] But the method of solution adopted in those cases are unsatisfactory. The problem to find a right-angled triangle having a given area occurs in an anonymous Greek manuscript[2] and its solution given therein is superior to that given by Mahāvīra.

Pairs of Rectangles.

Mahāvīra found two rectangles, (*i*) whose perimeters are equal but the area of one is double that of the other, or (*ii*) whose areas are equal but the perimeter of the one is double that of the other, or (*iii*) the perimeter of one is double that of the other and the area of the latter is double that of the former. But his method of solution is of a general character.

If (x, y) and (u, v) represent the sides of two rectangles, these problems are equivalent to the solution of the equations,

$$\left.\begin{aligned} m(x+y)&=n(u+v), \\ pxy&=quv. \end{aligned}\right\} \qquad (A)$$

Assume, says Mahāvīra[3]

$$y=s\ \{(\text{ratio of perimeters})\times(\text{ratio of areas})\},^2 \qquad \ldots \quad (1)$$

and $$x=y-1, \quad \text{if } p=q; \qquad \ldots \quad (2)$$

or $$x=3\ \{y-\overline{s\ (\text{ratio of areas})-1}\}, \text{ if } p\neq q, \qquad (2')$$

where s is an arbitrary multiplier, and the ratios are to be so presented as always to remain greater than or equal to unity.

Suppose $m\geq n$, $q\geq p$; then we shall have to assume

$$\left.\begin{aligned} y&=s\,\frac{m^2q^2}{n^2p^2}, \\ x&=3\left(s\,\frac{m^2q^2}{n^2p^2}-s\,\frac{q}{p}+1\right). \end{aligned}\right\} \qquad \ldots \quad (3)$$

[1] *Ibid*, vii. 152 ff.

[2] This manuscript has been published with German translation by J. L. Heiberg and comments by H. G. Zeuthen; "Einige griechische Aufgaben der unbestimmten Analytik," *Bibliotheca Mathematica*, VIII (3), 1907/8, pp. 118-34; also compare Heath, *History of Greek Mathematics*, 2 vols., Oxford, 1921, II, p. 446.

[3] *Gaṇita-sāra-saṁgraha*, vii. 131½-133. Raṅgācārya failed to grasp the real significance of the rule as he observes, "The solution given in the rule seems to be correct only for the particular cases given in the problems in stanzas 134 to 136," which is not true as will be sufficiently clear from the detail workings given here.

Substituting these values in (A), we get

$$u+v=\frac{m}{n}(4s\frac{m^2q^2}{n^2p^2}-3s\frac{q}{p}+3), \qquad \ldots \quad (4)$$

$$uv=3s\frac{m^2q}{u^2p}\left(s\frac{m^2q^2}{n^2p^2}-s\frac{q}{p}+1\right)$$

Then $(u-v)^2=\frac{m^2}{n^2}\left\{\left(4s\frac{m^2q^2}{n^2p^2}-3s\frac{q}{p}+3\right)^2\right.$

$$\left.-12s\frac{q}{p}\left(s\frac{m^2q^2}{n^2p^2}-s\frac{q}{p}+1\right)\right\},$$

$$=\frac{m^2}{n^2}\left\{\left(4s\frac{m^2q^2}{n^2p^2}-\frac{9}{2}s\frac{p}{q}+3\right)^2-\frac{3}{4}s\frac{q}{p}\left(s\frac{q}{p}-4\right)\right\},$$

Now if the arbitary multiplier s be so chosen that

$$s\frac{q}{p}=4, \qquad \ldots \quad (5)$$

we shall have

$$u-v=\frac{m}{n}(4s\frac{m^2q^2}{n^2p^2}-\frac{9}{2}s\frac{q}{p}+3). \qquad \ldots \quad (6)$$

From (4) and (6) we get

$$\left.\begin{aligned} u&=\frac{m}{n}\left(4s\frac{m^2q^2}{n^2p^2}-\frac{15}{4}s\frac{q}{p}+3\right). \\ v&=\frac{3}{4}s\frac{m}{n}\frac{q}{p}. \end{aligned}\right\} \qquad (7)$$

Substituting the value of s from (5) in (3) and (7), we have finally the solution of (A), when $m\geq n$, $q\geq p$ as

$$y=4\frac{m^2q}{n^2p}, \qquad v=3\frac{m}{n},$$

$$x=3\left(4\frac{m^2q}{n^2p}-3\right), \qquad u=4\frac{m}{n}\left(4\frac{m^2q}{n^2p}-3\right). \qquad (8)$$

Mahāvīra has observed that "when areas (of the two rectangles) are equal [2] (*tulya-phale*), we are to assume

$$y = s\frac{m^2}{n^2},$$

$$x = s\frac{m^2}{n^2} - 1.$$

But as we shall presently show this restriction is *not necessary*.

Assume in general, when $m \geq n$, $q \geq p$,

$$\left.\begin{aligned} y &= s\,\frac{m^2q^2}{n^2p^2}, \\ x &= s\,\frac{m^2q^2}{n^2p^2} - 1. \end{aligned}\right\} \tag{9}$$

Then substituting in (A), we get

$$u+v = \frac{m}{n}\left(2s\,\frac{m^2q^2}{n^2p^2} - 1\right), \tag{10}$$

$$uv = s\frac{m^2q}{n^2p}\left(s\,\frac{m^2q^2}{n^2p^2} - 1\right).$$

Now

$$(u-v)^2 = \frac{m^2}{n^2}\left\{\left(2s\,\frac{m^2q^2}{n^2p^2} - 1\right)^2 - 4s\frac{q}{p}\left(s\frac{m^2q^2}{n^2p^2} - 1\right)\right\},$$

$$= \frac{m^2}{n^2}\left\{\left(2s\,\frac{m^2q^2}{n^2p^2} - s\frac{q}{p} - 1\right)^2 - s\frac{q}{p}\left(s\frac{q}{p} - 2\right)\right\}.$$

If the arbitrary multiplier s be so chosen that

$$s\,\frac{q}{p} = 2, \tag{11}$$

we shall have

$$u-v = \frac{m}{n}\left(2s\,\frac{m^2q^2}{n^2p^2} - s\frac{q}{p} - 1\right). \tag{12}$$

From (10) and (12) we get

$$u = \frac{m}{n}\left(2s\,\frac{m^2q^2}{n^2p^2} - \frac{1}{2}s\frac{q}{p} - 1\right), \tag{13}$$

$$v = \frac{1}{2}s\frac{m}{n}\frac{q}{p},$$

Substituting the value of s from (11) we finally get another solution of (A) in the form

$$\left.\begin{aligned} y &= 2\,\frac{m^2 q}{n^2 p}, & u &= \frac{m}{n}, \\ x &= 2\,\frac{m^2 q}{n^2 p} - 1; & v &= \frac{m}{n}\left(4\frac{m^2 q}{n^2 p} - 2\right). \end{aligned}\right\} \quad (14)^{1}$$

when $m \geq n$, $q \geq p$.

Similarly we can obtain other sets of solution of the equations (A), for other relations between m, n and p, q by starting with the assumptions suggested by Mahāvīra.

Two problems similar to the above, more particularly to (i) and (iii), occur in an anonymous Greek manuscript which is supposed to belong to the period between Euclid and Diophantus.[2] The solution of one given in that text is deducible from the solution *(14)* and the solution of the other is different. The method of solution indicated by Mahāvīra is different from that suggested by Zeuthen[3] to have possibly been followed by the Greek writer.

General Solutions.

The problem noted above is really indeterminate. The Greek writers have given only a single solution of each of its particular cases treated by them, whereas Mahāvīra's method leads to two solutions of the general problem. Attempts for general solutions have been made by several later writers, including M. Cantor, P. Tannery, and L. E. Dickson.[4] Other general solutions can be obtained easily by proceeding in a way slightly different from that indicated by Mahāvīra.

[1] These solutions will be obtained after slight reduction on starting also on the hypothesis,

$$y = s\,\frac{m^2 q^2}{n^2 p_2}$$

$$x = s\,\frac{m^2 q^2}{n^2 p^2} - 3\left(s\,\frac{q}{p} - 1\right)$$

The original text also admits of such an interpretation.

[2] *Vide supra*, Heath, *Greek Math.*, II, pp. 444 *f.*

[3] *Bibliothica Mathematica*, VIII (3), 1907-8, pp. 118-34.

[4] Dickson, *Numbers*, II, pp. 486 *f.*

Assume, when $m \geq n$, $q \geq p$,

$$y = r\frac{m^2 q^2}{n^2 p^2} + t$$

$$x = \frac{m^2 q^2}{n^2 p^2}\left(s\frac{q}{p} - r\right)$$

where r, s, t are arbitrary quantities.

Substituting in (A), we get

$$u + v = \frac{m}{n}\left(s\frac{m^2 q^3}{n^2 p^3} + t\right)$$

$$uv = \frac{m^2 q}{n^2 p}\left(s\frac{q}{p} - r\right)\left(r\frac{m^2 q^2}{n^2 p^2} + t\right)$$

whence, if s be so chosen that

$$s\frac{q}{p} = \frac{r}{t}\left(r\frac{m^2 q^2}{n^2 p^2} - r\frac{q}{p} + 2t\right)$$

we shall have

$$u - v = \frac{m}{n}\left(s\frac{m^2 q^3}{n^2 p^3} - 2r\frac{q}{p} + t\right).$$

Therefore a general solution of (A) is

$$\left.\begin{aligned}
y &= r\frac{m^2 q^2}{n^2 p^2} + t \\
x &= \frac{r m^2 q^2}{t n^2 p^2}\left(r\frac{m^2 q^2}{n^2 p^2} - r\frac{q}{p} + t\right) \\
v &= r\frac{mq}{np} \\
u &= \frac{m}{nt}\left(r\frac{m^2 q^2}{n^2 p^2} + t\right)\left(r\frac{m^2 q^2}{n^2 p^2} - r\frac{q}{p} + t\right)
\end{aligned}\right\} \quad (15)$$

Again starting with the assumption

$$y=r\,\frac{m^2q^2}{n^2p^2}-t$$

$$x=\frac{m^2q^2}{n^2p^2}\left(s\,\frac{q}{p}-r\right)$$

and proceeding in the same way and choosing s such that

$$st=r^2$$

we get another general solution

$$\left.\begin{aligned} y&=r\frac{m^2q^2}{n^2p^2}-t, & v&=\frac{m}{n}\left(r\frac{q}{p}-t\right),\\ x&=r\frac{m^2q^2}{n^2p^2}\left(\frac{rq}{tp}-1\right); & u&=r\frac{mq}{np}\left(\frac{rm^2q^2}{tn^2p^2}-1\right). \end{aligned}\right\}\quad(16)$$

Putting $n=p=1$, we get the general solutions of

$$m(x+y)=u+v$$

$$xy=n\ uv$$

in rational integers as

$$\left.\begin{aligned} y&=t(rm^2q^2+t), & v&=rtmq,\\ x&=rm^2q^2(rm^2q^2-rq+t); & u&=m(rm^2q^2+t)(rm^2q^2-rq+t) \end{aligned}\right\}\quad(15\cdot1)$$

and

$$\left.\begin{aligned} y&=t(rm^2q^2-t), & v&=mt(rq-t),\\ x&=rm^2q^2(rq-t); & u&=rmq(rm^2q^2-t). \end{aligned}\right\}\quad(16\cdot1)$$

For $m=1$, the solution (16·1) becomes

$$\left.\begin{aligned} y&=t(rq^2-t), & v&=t(rq-t),\\ x&=rq^2(rq-t); & u&=rq(rq^2-t). \end{aligned}\right\}\quad(16\cdot2)$$

Putting $t=1, r=1$, in (16·2) we obtain the Greek solution and with $t=1$, and $rq=w$, we get Dickson's solution.

Pairs of Rational Isosceles Triangles.

Mahāvīra has indicated how to find two rational isosceles triangles whose perimeters are related in a given proportion and whose areas are related in another given proportion.[1] If (s_1, s_2) and (Δ_1, Δ_2) denote the perimeters and areas of two rational isosceles triangles, such that

$$s_1 : s_2 = m : n, \quad \Delta_1 : \Delta_2 = p : q \tag{1}$$

where m, n, p, q, are known integers, then the triangles will be obtained, says Mahāvīra, from the rectangles formed from

$$6\,\frac{n^2p}{m^2q},\; 2\,\frac{n^2p}{m^2q}-1 \;\text{ and }\; 4\frac{n^2p}{m^2q}+1,\; 4\,\frac{n^2p}{m^2q}-2, \;\text{ if } m^2q<n^2p$$

when the dimensions of the first are multiplied by m and those of the second by n.

The dimensions of the isosceles triangle formed from the first set of *bīja* are

$$\text{equal side} = m\times\left\{\left(6\,\frac{n^2p}{m^2q}\right)^2+\left(2\frac{n^2p}{m^2q}-1\right)^2\right\}$$

$$\text{base} = m\times 24\,\frac{n^2p}{m^2q}\left(2\,\frac{n^2p}{m^2q}-1\right)$$

$$\text{altitude} = m\times\left\{\left(6\,\frac{n^2p}{m^2q}\right)^2-\left(2\,\frac{n^2p}{m^2q}-1\right)^2\right\}$$

and from the second set

$$\text{equal side} = n\times\left\{\left(4\,\frac{n^2p}{m^2q}+1\right)^2+\left(4\,\frac{n^2p}{m^2q}-2\right)^2\right\}$$

$$\text{base} = n\times 4\left(4\,\frac{n^2p}{m^2q}+1\right)\left(4\,\frac{n^2p}{m^2q}-2\right)$$

$$\text{altitude} = n\times\left\{\left(4\frac{n^2p}{m^2q}+1\right)^2-\left(4\frac{n^2p}{m^2q}-2\right)^2\right\}$$

It can be easily verified that the perimeters and areas of the isosceles triangles thus obtained satisfy the conditions (1).

In particular, putting $m=n=p=q=1$, we have two isosceles triangles of sides, bases and altitudes (29, 40, 21) and (37, 24, 35) which have equal perimeters (98) and equal areas (420). This particular case was treated by Frans van Schooten the Younger (1657),[2] J. H. Rahn (1697)[3] and others.[4] Van Schooten's method seems to be different.

[1] *Gaṇita-sāra-saṁgraha*, vii. 187.

[2] Frans Van Schooten, *Exercitationum Mathematicarum libri V*, Amsterdam, 1656-7. *Cf.* Dickson, *Numbers*, II, p. 201.

[3] J. H. Rahn, *Algebra*, Zurich, 1659.

[4] Dickson, *Numbers*, II, p. 201.

Pairs of Isosceles Trapeziums.

Mahāvīra has shown how to find an isosceles trapezium equal in area and altitude to another isosceles trapezium whose sides are known.[1] Let a, b, c, h denote respectively the top-side, base, equal sides and the altitude of the known isosceles trapezium and let a', b', c', h denote the corresponding quantities of the required isosceles trapezium. Then since the two trapeziums are equal in area and altitude, we must have

$$b'+a'=b+a \tag{1}$$

Again $$c'^2-\left(\frac{b'-a'}{2}\right)^2=h^2$$

or $$\{c'+(b'-a')/2\}\{c'-(b'-a')/2\}=h^2$$

whence $$c'-(b'-a')/2=r, \qquad c'+(b'-a')/2=h^2/r$$

where r is any rational number. Then

$$c'=\frac{1}{2}(h^2/r-r), \tag{2}$$

$$b'-a'=h^2/r-r \tag{3}$$

From (1) and (3), we get

$$b'=(b+a)/2+(h^2/r-r)/2 \tag{4}$$

$$a'=(b+a)/2-(h^2/r-r)/2 \tag{5}$$

If $a=4$, $b=14$, $c=13$, $h=12$, taking $r=10$, we shall have[2] $a'=34/5$, $b'=56/5$, $c'=61/5$.

It has been stated before that, if m, n, p, q are any rational numbers such that $m^2 \pm n^2 < p^2 \pm q^2$, we must have

$$a=(p^2-q^2)-(m^2-n^2)$$

$$b=(p^2-q^2)+(m^2-n^2)$$

$$c=m^2+n^2$$

$$h=2mn$$

[1] *Gaṇita-sāra-saṁgraha*, vii. 173½

[2] *Ibid*, vii. 174½.

Substituting these values in (2), (4), (5), we get the dimensions of the equivalent isosceles trapezium as

$$a'=(p^2-q^2)-(4m^2n^2/r-r)/2$$

$$b'=(p^2-q^2)+(4m^2n^2/r-r)/2$$

$$c'=(4m^2n^2/r+r)/2$$

If, $m^2 \pm n^2 > p^2 \pm q^2$, the sides of the two isosceles trapezium which are equal in area and altitude will be

$$a=(m^2-n^2)-(p^2-q^2),$$

$$b=(m^2-n^2)+(p^2-q^2),$$

$$c=p^2+q^2$$

$$a'=(m^2-n^2)-(4m^2n^2/r-r)/2$$

$$b'=(m^2-n^2)+(4m^2n^2/r-r)/2$$

$$c'=(4m^2n^2/r+r)/2$$

These two isosceles trapeziums will also have equal diagonals.

Triangles and Quadrilaterals having a given Circum-radius.

Māhavīra proposes to solve a remarkably interesting problem: To find all rational triangles and quadrilaterals that can be inscribed in a circle of given diameter (D). His method of solution is very simple and elegant.[1] Find a rational triangle or a cyclic quadrilateral; calculate the diameter of its circum-circle and divide the given diameter by it. Dimensions of the optionally chosen triangle or quadrilateral multiplied by this quotient will give the dimensions of the required figure of the type.

It has been found before that the sides of a rational triangle will be proportional to

$$m^2+n^2,\ p^2+q^2,\ (m^2-n^2)\pm(p^2-q^2)$$

and its altitude is proportional to $2mn$ or $2pq$, m, n, p, q being any rational numbers such that $mn=pq$. The diameter of the circle circumscribed about this triangle will be proportional to[2]

$$\frac{(m^2+n^2)(p^2+q^2)}{2mn}$$

[1] *Gaṇita-sāra-saṁgraha*, vii. 221½.

[2] The rule for finding the diameter of the circle circumscribing a triangle or a quadrilateral has been stated by Mahāvīra in vii. 218½.

Then, the sides of a rational triangle inscribed in a circle of diameter D, will be

$$\frac{D.\,2pq}{p^2+q^2},\quad \frac{D.\,2mn}{m^2+n^2},\quad D.\,2mn.\,\frac{(m^2-n^2)\pm(p^2-q^2)}{(m^2+n^2)(p^2+q^2)}$$

and its altitude

$$\frac{D(2mn)^2}{(m^2+n^2)(p^2+q^2)}$$

The dimensions of a rational inscribed quadrilateral, as stated by Mahāvīra, have been noted before (p. 281). The diameter of its circum-circle will be[1]

$$(p^2+q^2)(m^2+n^2)^2$$

Then, according to Mahāvīra, the sides of a rational quadrilateral inscribed in a circle of diameter D, will be

$$D\left(\frac{2mn}{m^2+n^2}\right),\quad D\left(\frac{m^2-n^2}{m^2+n^2}\right),\quad D\left(\frac{2pq}{p^2+q^2}\right),\quad D\left(\frac{p^2-q^2}{p^2+q^2}\right);$$

its diagonals will be

$$\left\{2pq(m^2-n^2)+2mn(p^2-q^2)\right\}\times\frac{D}{(m^2+n^2)(p^2+q^2)},$$

$$\left\{(m^2-n^2)(p^2-q^2)+4mnpq\right\}\times\frac{D}{(m^2+n^2)(p^2+q^2)}$$

and its area

$$\frac{1}{2}\,\frac{D^2}{(m^2+n^2)^2(p^2+q^2)}\,\{2pq(m^2-n^2)+2mn(p^2-q^2)\}\times\{(m^2-n^2)(p^2-q^2)+4mnpq\}$$

so that the sides, diagonals and the area are all rational.

Mahāvīra's problem for rational quadrilaterals inscribable in a given circle was attempted, with extension to polygons, by Euler (c. 1781) and H. Schubert (1905).[1] The former gave a construction to find a polygon of n sides inscribed in a circle of radius unity, such that the sides, all diagonals, and the area are rational. His expressions for the sides and diagonals of a quadrilateral are complicated, still rational; but they do not make the area rational. Schubert's solutions for the sides, diagonals and areas of a quadrilateral inscribed in a circle of given radius will be rational in a very special case.

[1] Dickson, *Numbers* II, pp. 219 ff.

Mahāvīra's Originality.

We shall now examine how far Mahāvīra is original in the matter of his investigations into the solution of rational triangles and quadrilaterals, and how far he is indebted to the works of others. It is not an easy task at all. But to form an accurate estimate of Mahāvīra's contribution in this branch of the science of mathematics such a critical examination will be necessary. Now the problems treated by Mahāvīra in that matter may be chiefly divided into three classes: (*i*) Problems the like of which are not known to have been treated by any anterior mathematician;—*e.g.*, problems on right triangles involving areas and sides, rational triangles and quadrilaterals having a given area or circum-diameter, pairs of isosceles triangles; (*ii*) problems the like of which occur in the work of his Hindu predecessor Brahmagupta, *e.g.*, right triangle, right triangles with a given leg, rational triangles and quadrilaterals; (*iii*) problems the like of which are found in the work of an anonymous Greek writer, *e.g.*, right triangles having a given area, pairs of rectangles. In respect of the problems of Class (*i*) Mahāvīra's originality is of course assured. And in this class are some remarkably interesting and difficult problems. Mahāvīra's solutions of some of the problems of Class (*ii*) are same as those of Brahmagupta and in case of a few others, the solutions noted by the former are clearly modification, improvement or generalisation of the results of the latter. So it is highly probable, or almost certain that Mahāvīra was influenced by the work of his distinguished Hindu predecessor. There is also another reason, stated a little below, to lead us strongly to such a conclusion. But here also it should be noted to his credit, Mahāvīra is not always a blind follower of Brahmagupta but has, to a certain extent at least, extended and improved upon his works. Was Mahāvīra similarly influenced by the Greek writers? Besides the similarity noticed in respect of the problems in Class (*iii*), a few other things may appear to be in support of such a conjecture. As we have already noted, there is a very striking resemblance between Diophantus and Mahāvīra on one point. The expression of the sides of a right triangle in the form m^2-n^2, $2mn$, m^2+n^2, Diophantus calls "forming a right-angled triangle from m, n." And expression of the sides and diagonal of a rectangle as m^2-n^2, $2mn$, m^2+n^2, Mahāvira calls "forming a rectangle from m, n." One of Mahāvīra's solution of a problem to find a pair of rectangles having specified relations between their perimeters and their areas, is a generalisation of the solution given in that anonymous Greek manuscript. Against these points of near contact and similarity to suggest a hypothesis of Greek influence in Mahāvīra's mathematics, are found others of dissimilarity. Indeed there are significant points of divergence between the work of the Hindu writer and that of his Greek predecessor. Solutions given by the

former are quite different from those given by the latter except perhaps in one instance in which, as just stated, Mahāvīra's solutions are far more general than those of the Greek writer. On the contrary in the matter of finding a right triangle having a given area, Mahāvīra's method is inferior. Their methods of solution in every case are different. Another noteworthy fact in this connexion is that Greek Diophantus appears to have been quite unaware of the work in that Greek manuscript.[1] How can it then be supposed that a distant foreigner like Mahāvīra (c. 850) might have been aware of it, coming moreover as he does nearly six centuries after Diophantus (c. 275)? In the matter of the similarity of certain expressions of Diophantus and Mahāvīra, it should be noted that the former always speaks of "forming a right-angled triangle," whereas the latter of "forming a rectangle." This difference cannot be considered immaterial. Further, some of the problems of Diophantus, especially those relating to rational right triangles whose area *plus* or *minus* the one or more of the sides is a given number, might well have been included by Mahāvīra, if he had been at all acquainted with the work of the former. Taking all these facts into consideration it can be assumed without any fear of opposition that there is no Greek influence in Mahāvīra's work in the branch of mathematics under discussion and that the points of similarity noticed between the works of these Hindu and Greek writers are matters of chance agreement. So the originality of Mahāvīra is established here again. In the works of Mahāvīra we generally notice the application of two principal methods for the solution of rational triangles and quadrilaterals: (1) one is the method of juxtaposition of two or more rational right triangles and (2) the other is a method analogous to the Rule of Single False Position in Algebra. The former is a very powerful method and its discovery is due to Brahmagupta. Here is then an additional proof of Brahmagupta's influence on Mahāvīra. The credit for the discovery of the other method which is no less important, is entirely Mahāvīra's own.[2]

[1] Heath, *Greek Math.*, p. 447. This manuscript (probably of the twelfth century) was discovered at Constantinople. It is supposed to be belonging to the period between Euclid and Diophantus because there is nothing common between it and Diophantus's *Arithmetica.*

[2] In a paper to be published shortly, entitled *On the Relation of Mahāvīra to Śrīdhara*, the present writer has proved conclusively that "Mahāvīra was not only well acquainted with, but was also greatly influenced by, the writings of Śrīdhara." But not one of those proofs refers to the topic under consideration here. The available treatise of Śrīdhara, *Triśatikā*, does not indeed treat of problems about the solution of rational triangles and quadrilaterals. But his lost work, of which the present one is admittedly a short abridgement, might have done it. Hence our conclusions about the originality of Mahāvīra in this matter must be subject to this reservation.

ZUR THEORIE DER AFFINMINIMALFLÄCHEN

VON

E. SALKOWSKI (*Berlin*).

(*Communicated by Professor Ganesh Prasad.*)

Die von *Darboux*[1] entdeckten und von *W. Blaschke* sogenannten und eingehend untersuchten Affinminimalflaechen[2] gehören zu den einfachsten und reizvollsten Gebilden der affinen Differentialgeometrie. Wenn auch der Formelapparat, der ihre Theorie beherrscht, ziemlich vollständig und gebrauchsfertig vorliegt,[2] so dürfte es sich doch wohl verlohnen, die Untersuchung noch einmal, und zwar von einem wesentlich geometrischen Standpunkte aus aufzunehmen, von dem sich alle Einzelheiten besonders über-sichtlich anordnen und nicht, wie bei analytischem Vorgehen in der Regel, als zufällige und überraschende Ergebnisse der Rechnung erscheinen. Dass dabei auch manche bisher übersehene Einzelheiten die zu einem abgerundeten Gesamtbilde gehören, gewonnen werden, braucht nicht besonders betont zu werden.

Der leichten Lesbarkeit halber sollen auch gelegentlich bekannte Dinge, die in dem Zusammenhang nicht entbehrt werden konnen, wenigstens kurz auseinandergesetzt werden. Der eigentlichen Untersuchung sind noch einige Tatsachen aus der Theorie der Netzflächen und der Gewindekurven, die später benutzt werden, vorangestellt.

I. *Über die Asymptotenlinien der Netzflächen.*

Auf einer jeden geradlinigen Fläche geht durch einen regulären Punkt P eine Asymptotenlinie, die nicht mit der Erzeugenden zusammenfällt. Sie ist, sofern sie nicht selbst geradlinig ist, dadurch gekennzeichnet, dass ihre Schmiegungsebene mit der Tangentialebene zusammenfält. Man erhält die Richtung ihrer Tangente, wenn man durch den Punkt P diejenige Gerade legt, die zwei benachbarte[3] Erzeugende der Fläche schneidet. Legt

[1] *Darboux*, Theorie des surfaces III, S. 368.

[2] Vgl. die zusammenfassende Darstellung von *W. Blaschke* und *K. Reidemeister* im 2. Bande der Vorlesungen über Differentialgeometrie (Berlin, 1923).

[3] Selbstverständlich soll dieser kurze Ausdruck nur den bekannten Grenzprozess kennzeichnen, durch den die Gerade erhalten wird.

man an die Fläche diejenige F_2, die mit ihr die Erzeugende p des Punktes P und die beiden Nachbarerzeugenden[1] gemeinsam hat (die *Lie*sche F_2 der Fläche für die Gerade p), so sind die Richtungen aller Asymptotenlinien in ihren Schnittpunkten mit p durch die Geraden Linien der zweiten Schar dieser *Lie*schen Schmiegungsflache gegeben.[2] Aus den projektiven Eigenschaften der Linienflächen 2. Ordnung folgt hieraus sofort bekannte Satz von *Chasles :*

Die Erzeugenden einer Linienfläche werde von vier Asymptotenlinien in konstantem Doppelverhältnis geschnitten.

Unter einer *Netzfläche* versteht man bekanntlich eine geradlinige Fläche, deren Erzeugenden einem Liniennetz, d. h. einer linearen Kongruenz, angehören.[3] Hier sind auch zwei der nicht zu den Erzeugenden gehorigen Asymptotenlinien geradlinig, die beiden " Leitgeraden" des Netzes und der Fläche (Für die anschauliche Behandlung setzen wir das Netz als hyperbolisch, die beiden Leitgeraden als reell und verschieden, voraus).

Diese Leitgeraden und ein beliebiges anderes Paar von weitem krummen Asymptotenlinien bestimmen daher auf der Erzeugenden ein konstantes Doppelverhältnis Projiziert man die eine Leitgerade in das Unendliche und die zweite senkrecht zur Stellung der ersten, so geht die Netzfläche in ein gerades Konoid über.

Die Netzflächen sind den Geraden Konoiden projektiv äquivalent.

Für ein Konoid geht das Doppelverhältnis in ein Teilverhältnis über, so dass hier der Satz von *Chasles* die Form annimmt :

Auf einem Konoid wird die Strecke, die zwei krumme Asymptotenlinien auf der Erzeugenden abschneidet, von der Achse in konstantem Verhältnis geschnitten. Projiziert man daher die Asymptotenlinien eines Konoids in der Richtung der Achse auf eine beliebige Ebene, so sind die Bilder ähnliche und ähnlich liegende Kurven. Diese Tatsache, die sich übrigens aus der auf Asymptotenlinien bezogenen Darstellung der Fläche.[4]

$$x_1 = -u\phi'(v),\ x_2 = uv\ \ x_3 = v\phi'(v) - 2\phi(v)$$

unmittelbar ablesen lässt, bedeutet, *dass alle Asymptotenlinien desselben*

[1] Auch hier dürfte ein Missverständnis dieses streng "sinnlosen" Ausdruckes nicht zu befurchten sein.

[2] *W. Schell.* Allgememeine Theorie der Kurven doppelter Krümmung. 3. Aufl. (1914) S. 80.

[3] *H. Mohrmann.* Über die Haupttangentenkurven von Netzflächen. Math. Ann. 73 (1912) S. 571.

[4] *Bianchi-Lukat.* Differentialgeometrie (Leipzig, 1910) S. 133. Durch eine naheliegende Aenderung in der Bezeichnung ist in der obigen Formel das bei Bianchi auftretende Integralzeichen vermieden.

Konoids affin äquivalent sind. Jede von ihnen gehört einem der ∞^1 Gewinde[1] an. In der Tat ist für u=konst.

$$x_1 x'_2 - x_2 x'_1 = u^2 x'_3.$$

Da die Gewindekurven für gewisse ausgezeichnete Klassen von Affinmininalflächen eine besondere Rolle spielen, seien ihre Eigenschaften, so weit sie für das folgende von Bedeutung sind, hier zusammengestellt. Dabei bietet sich die Gelegenheit, die Ergebnisse einer früheren Arbeit [2] zu ergänzen.

II. Gewindekurven.

Wir bezeichnen in der Folge eine Kurve U durch ihren Radiusvektor $\mathfrak{U}$, der als Funktion des Affinbogens u vorrausgesetzt ist, ihre Koordinaten, die Masszahlen der Komponenten des Radiusvektors in der Richtung der Koordinatenachsen 1, 2, 3 werden, wenn sie überhaupt gebraucht werden, durch die entsprechenden, der Vektorbezeichnung abgehängten Indizes, also mit U_1, U_2, U_3 bezeichnet. Akzente bezeichnen stets Ableitungen nach dem Affinbogen der Kurve. Dementsprechend wird eine Fläche $x(u, v)$ durch den Radiusvektor bezeichnet, die Koordinaten eines Flächenpunktes durch x_1, x_2, x_3 ; x_u, x_v bedeuten die abgeleiteten Vektoren, deren Komponenten $\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}$ bezw. $\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}$ sind. In der Regel wird man bei Flächen darauf verzichten müssen, u und v als Affinbogen der Parameterkurven vorauszusetzen.

Die natürliche Gleichung einer Gewindekurve ist durch die beiden Affinkrümmungen

$$k(u) = (U'', U''', U^{IV}), \qquad t(u) = -(U'', U''', U^{IV})$$

in der Form

$$\text{(I)} \quad k' - t = 0$$

gegeben.[3] Für eine solche Kurve ist der Vektor

$$\text{(II)} \quad U''' + kU' = \text{konst.}$$

[1] *H. Mohrmann*, a. a. O

[2] Leipz. Ber. 70 (1918) S. 160.

[3] Vgl. neben der unter 9) genannten Originalarbeit die Darstellung in *Blaschke* Differentialgeometrie II, S. 88 ff.

und gibt die Richtung des Durchmessers des Gewindes, dem die Tangenten der Kurve U angehören.[1]

Zieht man durch jeden Punkt einer Gewindekurve einen Schmiegungsstrahl (d. h. eine Gerade in der Schmiegungsebene), der eine feste Gerade schneidet, so gehören diese Gerade auch dem Liniengebüsch (speziellen Gewinde) an, dessen Achse die feste Gerade ist, sie bilden also eine Netzfläche. Sind insbesondere die Schmiegungsstrahlen einer festen Ebene parallel, so bilden sie ein Konoid, d. h. sie schneiden alle noch eine zweite, und zwar eigentliche feste Gerade, die der Stellung der Ebene konjugiert ist d. h. *die Achse des Konoids ist eine Durchmesser des Gewindes.*

Das *Liesche* Paraboloid, das durch drei benachbarte Geraden des Konoids bestimmt ist, hat als zweite Geradenschar die Tangenten an die krummen Asymptotenlinien der Flaeche. Auch diese Geraden muessen eine feste Richtebene besitzen; sie muessen naemlich alle die unendlich ferne Gerade des Lieschen Paraboloid, die durch die unendlich fernen Punkte der Gewindekurven und der Achse des Gewindes bestimmt ist, schneiden. Dies ist aber nichts anders als die Durchmesserebene, durch die Tangente der Gewindekurve, eine Ebene, die durch die Vektoren U′ und U‴ bestimmt ist. Bemerkenswert ist, dass die Stellung dieser Richtebene nur von der Kurve abhängig ist, nicht von der Auswahl der Schmiegungsstrahlen.

Zieht man durch die Punkte einer Gewindekurve (P) die Schmiegungsstrahlen, die einer beliebigen festen Ebene parallel sind, so sind auf allen ∞^2 so entstehenden Konoiden die Richtungen des Asymptotenlinien in den Punkten einer Erzeugenden p parallel zu der Durchmesserebene des zur Geraden p gehoerenden Punktes der Kurve (P).

Fuehrt man dieselben Konstruktion fuer eine beliebige Raumkurve durch die also keine Gewindekurve sein soll, so kann man jedenfalls immer durch vier benachbarte einer festen Ebene parallele Schmiegungsstrahlen eine

[1] Dass der Vektor auf der linken Seite der Gleichung (II) konstant ist, ergibt eine einfache Differentiation, wobei die grundlegende Beziehung $U^{IV}+kU''+tU'=0$ und die Gleichung 1) zu berucksichtigen ist. Seine geometrische Bedeutung, die nach einer Bemerkung von *Blaschke* (Diff.—G. II, S. 99) wohl bereits in einer mir nicht zugänglichen Arbeit von Čech angegeben worden ist, folgt bei Spezialisierung des Koordinatensystems durch eine ganz einfache Rechnung. Setzt man namlich die *Liesche* Gleichung der Gewindekurven in der Form an $U'_1 U_2 - U'_2 U_1 = aU'_3$, so ergibt sich durch wiederholte Differentiation dieser Gleichung $U'''_1 U'_2 - U'''_2 U'_1 = 0$, die Determinante zweier in (2) steckenden Gleichungen: $U'''_2 + kU'_2 = c_2$, $U'''_3 + kU'_3 = c_1$. Diese Gleichungen koennen nur bestehen, wenn $c_1 = c_2 = 0$, der Vektor U also parallel zur Achse des gegebenen Gewindes wird.

Gerade Linie legen und so ein Konoid konstruieren, das die Flaeche der Schmiegungstrahlen in diesen vier Nachbarerzeugenden beruehrt. Der eben ausgesprochene Satz gilt daher fuer eine beliebige Kurve, mit der Abaenderung, dass statt Konoid immer " geradlinige Flaeche mit Richtebene " gesetzt wird.

III. Die Affinminimalflaechen : Erzeugungsweise.

Wir gehen von der Darbouxchen Erzeugungsweise [1] aus :

Man bestimme in jedem Punkte einer Schubflaeche (Translationsflaeche) die Schnittlinie der Schmiegungsebenen der erzeugenden Kurven. Diese Geraden bilden eine Kongruenz, deren Brennflaechen zwei zugeordnete Affinminimalflaechen sind. Die Schubflaeche ist die Mittelflaeche dieser Kongruenz.

Wir setzen die erzeugenden Kurven als unebene Raumkurven voraus. Die Erzeugungsweise von *Darboux* setzt das Problem der Affinminimalflaechen mit einen auch sonst durch seine bemerkenswerten Eigenschaften ausgezeichneten geometrischen Zusammenhang.

Man verbinde alle Punkte P einer Raumkurve (P) mit allen Punkten Q einer Raumkurve (Q). Die Mittelflaeche x dieser Kongruenz ist eine Schubflaeche, deren erzeugende Kurven zu (P) und (Q) im Verhaeltnis $1:2$ aehnlich sind. Die Schnittlinien pq der Schmiegungsebenen der Kurven bilden eine Kongruenz, deren Brennflaechen die abwickelbaren Tangentenflaechen von (P) und (Q) sind, und die Huellflaeche der Ebenen, die in den Mittelpunkten der Brennstrecken pq parallel zu den entsprechenden Tangenten von (P) und (Q) gelegt sind, ist die Mittelflaeche x der ersten Kongruenz.

Zieht man durch die Punkte der Flaeche x die Parallelen g zu den entsprechenden Strahlen pq, so entsteht die Kongruenz, deren Brennflaechen die Affinminimalflaechen sind.

Laengst bekannt ist ein metrischer Sonderfall dieses Sachverhalts. Sind naemlich (P) und (Q) isotrope Kurven, so ist x eine Minimalflaeche in gewoehnlichen Sinne ; die Geraden pq bilden eine isotrope Kongruenz, ihre Erzeugenden gehoeren den Schmiegungsebenen sowohl von P *als* auch von Q an, stehen *also* auf den Tangenten beider Kurven senkrecht. Die Parallelen zu pq durch die Punkte von x sind daher die Normalen der Minimalflaeche.

Zu den Affinminimalflaechen gehoeren also auch die Evoluten der gewoehnlichen Minimalflaechen.

Nach diesem Exkurs, der uns eine besondere Klasse von Affinminimalflaechen kennen lehrte und zeigte, dass reelle Flaechen auch dann entstehen

koennen, wenn die Schubkurven (P) und (Q) imaginaer sind, gehen wir weiter in der Analyse des allgemeinen geometrischen Sachverhalts.

Haelt man P fest und laesst Q die Kurve (Q) durchlaufen, so beschreibt x auf der Schubflaeche eine Kurve V, pq umhuellt auf der festen Schmiegungsebene π des Punktes P eine Kurve, den Schnitt der Ebene π mit der Tangentenflaeche der Kurve (Q). Die entsprechenden Geraden g sind also alle parallel zur Ebene π, beschreiben somit eine Regelflaeche mit Richtebene. Da sie alle in den Schmiegungsebenen der Kurve V liegen, d. h. Schmiegungsstrahlen der Kurve V sind, ist diese eine Asymptotenlinie der Regelflaeche. Genau eben ist natuerlich die Kurve U eine Asymptotenlinie auf der Linienflaeche derjenigen Strahlen g, die sie schneiden.

Die Kurven U, V der Schubflaeche zerlegen die Geraden der Kongruenz der Geraden g in zwei Scharen von Regelflachaechen mit Richtebene auf denen die Leitkurven U, bezw. V Asymptotenlinien sind. Die Brennflaechen der Kongruenz (g) lassen sich demnach auf doppelte Weise durch Umhuellung von je ∞^1 Regelflaechen mit Richtebene erzeugen.

IV. Analytische Darstellung.

Die Kurven (P) und (Q) seien durch die Vektoren 2U und 2V dargestellt, sodass die Schubflaeche x durch die Gleichung

$$x = \mathrm{U} + \mathrm{V} \tag{1}$$

gegeben ist. Dabei sei u der Affinbogen der Kurve U, v diejenige von V so dass die Gleichungen

$$(\mathrm{U}'\mathrm{U}''\mathrm{U}''') = 1 \qquad (\mathrm{V}'\mathrm{V}''\mathrm{V}''') = 1$$

bestehen. Dabei ist bekanntlich der Affinbogen nur fuer positiv gewundenen Kurven reell, bei negativ gewundenen dagegen imaginaer.

Die Gerade g gehoert der Schmiegungsebene von U und der von V an, so dass fuer jeden ihrer Punkte

$$(\phi - \chi, \ \mathrm{U}'\mathrm{U}'') = 0 \qquad (\phi - \chi, \ \mathrm{V}'\mathrm{V}'') = 0$$

sein muss, oder, wenn man die Vektorprodukte

$$[\mathrm{U}'\mathrm{U}''] = \xi \qquad [\mathrm{V}'\mathrm{V}''] = \eta \tag{2}$$

setzt, $\quad (\phi - \chi, \ \xi) = 0 \qquad (\phi - \chi, \ \eta) = 0$

Somit wird die Kongruenz (g) durch die Gleichung

(3) $$\phi=\chi+t\zeta$$

gegeben, wobei

(4) $$\zeta=[\xi\eta]$$

einen Vektor bedeutet, der den Schmiegungsebenen der beiden Kurven U und V angehoert, also sowohl in der Form

(5) $$\zeta=\alpha U'+\beta U''$$

als auch

(5[1]) $$\zeta=\gamma V'+\delta U''$$

angesetzt werden kann. In der Tat werden nach bekannten Formeln der Vektorrechnung[12]

(6) $$\alpha=-(U''V'V'')=-\beta_u \qquad \beta=(U'V'V'')$$

(6[1]) $$\gamma=(U'U''V'')=-\delta_v \qquad \delta=-(U'U''V').$$

Fuer einen Punkt der Brennflaeche muss der Vektor $d\phi$ parallel zu werden;

$$d\phi=\zeta d\lambda,$$

so dass $$d\chi+td\zeta+(dt-d\lambda)\zeta=0$$

oder $$(\chi_u+t\zeta_u)du+(\chi_v+t\zeta_v)dv+(dt-d\lambda)\zeta=0$$

ist. Die beiden ersten Klammern und ζ sind also drei komplanare Vektoren, so dass ihre Determinante

$$(U'+t\zeta u,\ V'+t\zeta v,\ \zeta)=0$$

sein muss. Diese Gleichung ist in t quadratiert, also von der Form

$$Pt^2+Qt+R=0$$

wobei $$P=(\zeta_u\ \zeta_v,\ \zeta)$$

$$Q=(U',\ \zeta_v,\ \zeta)+(\zeta_u V'\zeta)$$

$$R=(U'V'\zeta)$$

gesetzt ist. Da ζ der Schmiegungsebene von U angehoert und bei der Veraenderung von V allein ihr parallel bleibt, ist die Determinante

$$(U'\zeta_v\zeta)=0$$

und entsprechend wird $$(\zeta_u V'\zeta)=0$$

[12] Vgl. z. B. *Lagally*, Vektorrechnung (Leipzig 1928), S. 33.

so dass Q verschwindet. Durch Rechnung folgt aus den Formeln

$$\zeta_u = \gamma_u V' + \delta_u V''$$

$$\zeta_v = \alpha_v U' + \beta_v U''$$

dasselbe Ergebnis, das geometrisch besagt, dass *die Schubflaeche x* die Mittelflaeche der Kongruenz (3) ist. Desgleichen findet man

$$P = R = \beta\delta$$

durch einfache Ausrechnung, so dass fuer die Brennpunkte des Strahls

$$t^2 = 1$$

sein muss.

Damit erhaelt man fuer die Gleichungen der beiden Brennflaechen

$$\text{(A)} \quad \phi = U + V + i\zeta$$

$$\text{(A}^1\text{)} \quad \bar{\phi} = U + V - i\zeta$$

Gleichungen, die dann reell sind, wenn die Kurven U, V entgegengesetztes Vorzeichen der Torsion haben.

Zur folgenden werden wir uns in der Regel nur mit der einen Brennflaeche (A) zu befassen haben.

V. Asymptotenlinien der Affinminimalflaechen.

Betrachten wir zunaechst die Linienflaeche der der Kurve v=konst. entsprechenden Regelflaeche der Geraden (g):

$$(7) \quad \phi = U + V + w[\xi\eta] \qquad (v = \text{konst.})$$

Dann wissen wir aus den Ueberlegungen des Abschnitts III, dass die Tangenten an saemtliche Asymptotenlinien fuer alle Punkte der Geraden u=Konstant zu einer Ebene parallel sein muessen, und zwar parallel der durch den Tangentenvektor U′ im Punkte $x = u + v$ bestimmten Durchmesserebene des Schmiegungsgewindes; sie muessen sich demnach durch die Vektoren U′ und U‴ linear darstellen lassen. Nun ist

$$\phi_u = U' + w\xi_u = U' + w(\alpha_u U' + \beta U''')$$

$$\phi_w = \alpha U' + \beta U'''$$

Eine Richtung

$$d\phi = \phi_u du + \phi_w dw$$

ist daher nur dann von U‴ unabhaengig, wenn dw=konst. ist. Daraus folgt ohne weitere Rechnung, dass die Kurven w=konst. die Asymptotenlinien auf der Flaeche (1) sind.

Einen zweiten, hiervon unabhaengigen Nachweis fuer diese Tatsache erhaelt man, wenn gezeigt werden kann, dass die Schmiegungsebene der Kurve w=konst in der Tangentialebene der Flaeche liegt, d.h. dass der Vektor ϕ_{uu} mit ϕ_u und ϕ_w komplanar ist.

Nun ist

$$\phi_u = U' + w\xi_u = U' + w[\xi'\eta]$$

$$\phi_{uu} = U'' + w\xi_{uu} = U' + w[\xi''\eta].$$

Da aber

$$\xi = [U'U'']$$

gesetzt war, also

$$\xi' = [U'U''']$$

und

$$\xi'' = [U''U'''] - k[U'\ U'']$$

ist, so wird

$$[\xi\xi'] = \Big[\,[U'U'']\,[U'U''']\,\Big] = U'$$

$$(8) \quad [\xi\xi''] = \Big[\,[U'U'']\,[U''U''']\,\Big] = U''$$

$$[\xi'\xi''] = \Big[\,[U'U''']\,[U''U''']\Big] - k\,\Big[\,[U'U''']\,[U'U'']\Big] = U''' + kU'$$

Damit wird:

$$\phi_u = [\xi', w\eta - \xi]$$

$$(9) \quad \phi_w = [\xi, \eta] = \frac{1}{w}\,[\xi, w\eta - \xi]$$

$$\phi_{uu} = [\xi'', w\eta - \xi],$$

d h. die drei Vektoren $\phi_u \phi_{uu} \phi_w$ sind, da sie zu demselben Vektor $(w\eta - \xi)$ senkrecht stehen, komplanar.

Insbesondere kommt man fuer $w = i$ auf die Kurve, in der die Linienflaeche (7) die Brennflaeche beruehrt, da diese Kurve auf der Linienflaeche Asymptotenlinie ist, muss sie auch auf der Brennflaeche eine solche sein.

Den Schubkurven der Translationsflaeche entsprechen auf den Affinminimalflaechen die Asymptotenlinien.

Betrachtet man einen zweiten Kurve v_1=konst. so besitzt die entsprechende Flaeche (g) zwar eine andere Richtebene; denkt man sie aber parallel zu sich selbst verschoben so dass die Leitkurven u zusammenfallen, so erhaelt man zwei aus Schmiegungsstrahlen derselben Kurve bestehende Flaechen, deren Liesche Paraboloid fuer die Geraden der zweiten Schar dieselbe Richtebene aufweisen, und zwar eine Durchmesserebene des Schmiegungsgewindes (vgl. den Schluss des Abschnitts II).

Da bei der Verschiebung der Kurve U aus der Lage $v=v_0$ in die Lage $v=v_1$ die Stellung der Ebene keine Aenderung erfaehrt, so sind die Tangenten an alle Asymptotenlinien u laengs einer Kurve v derselben Ebene parallel, und zwar ist deren Stellung durch die Vektoren U' und U''' bestimmt.

Auf jeder Affinminimalflaeche sind die Tagenten an die Asymptotenlinien der einen Schar im Schnittpunkte mit einen festen Asymptotenlinien der zweiten Schar einer festen Ebene parallel.

Analytisch ergibt sich der Sachverhalt aus den Ausdruecken

$$\phi_u = U' + i[\xi'\eta] = [\xi'.\ i\eta - \xi] \tag{10}$$
$$\phi_v = V' + i[\xi\eta'] = i[\eta', i\eta - \xi].$$

Denn hieraus folgt

$$(\phi_u \xi') = 0$$

d.h.

$$(\phi_u U'U''') = 0$$

und zwar, wie aus (9) und (6) hervorgeht:

$$\phi_u = U'(1 - i\beta_{uu}) + \beta U'''$$

Entsprechend ist auch

$$(\phi_v V'V''') = 0$$
$$\phi_v = (1 - i\delta_{vv})V' + \delta V'''.$$

Wir koennen aus dem eben bewiesenen Satze gleich eine einfache Anwendung auf ein metrisches Problem ablesen, indem wir fragen:

Welche Affinminimalflaechen sind auch Minimalflaechen im gewoehnlichen Sinne?

Die gewoehnlichen Minimalflaechen sind dadurch gekennzeichnet, dass ihre Asymptotenlinien ein Orthogonalsystem bilden. Daher sind die Tangenten Kurven (u) laengs einer festen Kurve (v) auf den gesuchten Flaechen Normalen der Kurve v, und zwar, da sie in der Schmiegungsebene liegen (die Kurve (v) ist Asymptotenlinie:) sind sie Hauptnormalen. Nach dem eben bewiesenen Satze muessen aber diese Hauptnormalen alle einer festen Ebene parallel sein. Derartige Kurven aber sind allgemeine Schraubenlinien.[13]

Die Flaechen, die zugleich gewoehnliche und affin Minimalflaechen sind, haben zwei Scharen von allgemeinen Schraubenlinien zu Asymptotenlinien.

Das Gausssche Bild einer Asymptotenlinie faellt bekanntlich mit dem Bilde ihrer Binormalen zusammen; das Binormalenbild einer Schraubenlinie

Schell, Theorie der Kurven (1914), S. 11.

aber ist ein Kreis.[14] Da das Gausssche Bild des Systems von Asymptotenlinien einer Minimalflaeche bekanntlich ein sphaerisches Orthogonalsystem ist, so ergibt sich das von Thomsen [15] rechnerisch abgeleitete Ergebnis:

Das Gausssche Bild der Asymptotenlinien der gesuchten Flaechen ist ein System sich senkrecht schneidendes Kreis scharen.

Die explizite Bestimmung dieser Minimalflaechen gehoert zu den klassischen Aufgaben der Differentialgeometrie.

VI. Die Affinnormale der Affinminimalflaechen.

Die Affinnormale einer Flaeche in einem regulaeren Punkte ist die Verbindungslinie der Flaechenpunktes mit dem Mittelpunkte der *Lie*schen *F2*. Fuer die affinen Minimalflaechen ist die *Lie*sche *F2* ein Paraboloid, die Affinnormale ist daher die Schnittlinie der Richtebenen der beiden Scharen von Erzeugenden dieses Paraboloids. Nun waren diese Richtebenen im vorigen Abschnitt bereits bestimmt, und zwar als Durchmesserebenen der Schmiegungsgewinde der Parameterkurven U und V. Man erhaelt also die Affinnormalen im Punkte ϕ, wenn man durch ihn die Parallele zu der Schnittlinie der Ebenen $\lambda\ U'+\mu U'''$ bzw. $\bar{\lambda}\ V'+\bar{\lambda}\ V'''$ konstruiert.

Dieser Sachverhalt ergibt sich auch durch einfache Rechnung. Fuer eine auf Asymptotenlien bezogene Flaeche wird der—zweckmaessig normierte—Affinnormalevektor $\boldsymbol{y}$ durch die Gleichung

$$\boldsymbol{y}=\frac{1}{\mathrm{F}}\,\phi_{vv}$$

gegeben, wobei.

$$\mathrm{F}^2=(\phi_u\phi_v\phi_{uv})$$

gesetzt ist. Fuer die Affinminimalflaeche (A) waren

$$\begin{aligned}(10)\quad \phi_u&=[\xi',\ i\eta-\xi]\\ \phi_v&=i[\eta',\ i\eta-\xi].\end{aligned}$$

woraus

$$(11)\quad \phi_{uv}=i\zeta_{uv}=i[\xi'\eta']=i\left[\,[\mathrm{U'U'''}][\mathrm{V'V'''}]\,\right].$$

folgt. Fuehrt man auch noch den normierten Kontravarianten "Normalvektor" H durch die Gleichung

$$\mathrm{FH}=[\phi_u\phi_v]=i(\xi',\ \eta',\ i\eta-\xi)(i\eta-\xi)$$

ein, so wird

$$\mathrm{H}\boldsymbol{y}=1$$

[14] *Schell*, Theorie der Kurven (1914), S. 25.

[15] *Thomsen*, Hamburg, Abh. *2* (1923), S. 69. Blaschke, Differentialgeometrie, II S, 187,

somit

$$F^2 = -(i\eta-\xi, \xi', \eta)^2,$$

also schliesslich

$$F = i(i\eta-\xi, \xi', \eta'). \tag{12}$$

Damit hat man

$$y = \frac{\left[[U'U]'''[V'V'''] \right]}{(i\eta-\xi, \xi'\eta')}, \tag{13}$$

waehrend

$$H = i\eta-\xi, \tag{14}$$

eine Gleichung, die nur die bekannte Tatsache ausdrueckt, dass der Normalvektor einer Affinminimalflaeche eine zu dieser Flaeche kontravariante Schubflaeche beschreibt. [16]

Der Vektor

$$y = \frac{i}{F}\left[[U'U'''] [U'U'''] \right]$$

laesst sich sowohl durch U', U''', als auch durch V', V''' darstellen, und zwar wird

$$Fy = i(\alpha_{u\,v} U' + \beta_v U''')$$

$$Fy = (i\gamma_{u\,v} U' + \delta_u V''').$$

Es bilden also laengs einer Asymptotenlinie u=konst. die Affinnormalen eine geradlinige Flaeche, die eine Richtebene parallel U', U''' besitzt; entsprechendes gilt fuer die Asymptotenlinien v=konst. Ein bemerkenswerter Sonderfall liegt vor, wenn die U Kurven und V Kurven zweier Gewinde mit parallelen Achsen besitzen: hier ist die Schnittlinie der Durchmesserebenen immer ein Durchmesser; alle Affinnormalen dieser Flaeche sind zu einander parallel: die Affinminimalflaeche ist also eine "uneigentliche Affinsphaere."[17]

Eine Schubflaeche, deren erzeugende Kurven zwei Gewinden mit parallelen Durchmesser angehoeren, fuehrt auf uneigentliche Affinsphaeren.

Da der in diesem Falle vorliegende Sachverhalt geometrisch und besonders reizvoll ist, sei ihm noch eine besondere Untersuchung gewidmet.

VIII. *Gewindekurven und uneigentliche Affinsphaeren.*

Die eine Kurve U schneidenden Geraden g bilden eine Flaeche von Schmiegungsstrahlen von U, die als Richtebene eine feste Schmiegungsebene

[16] *Blaschke*, Differentialgeometrie, II, S. 178.

[17] *Blaschke*, Differentialgeometrie, II,

von (V) besitzt. Ist insbesondere U eine Gewindekurve, so wissen wir (Abschnitt II), dass die Flaeche ein Konoid sein muss, dessen Leitkurve parallel dem Durchmesser des Gewindes ist. Wir wissen ferner dass die saemtlichen Asymptotenlinien der Flaeche affin aequivalent zur Kurve U, also auch Gewindekurven sind. Daraus ergibt sich, dass gleichzeitig mit der Kurve U alle Asymptotenlinien der Affinminimalflaeche v=konst. Kurven auf Gewinden mit parallelen Durchmesser sein muessen.

Rechnerisch ueberzeugt man sich von der Richtigkeit dieser Schlussweise durch den Nachweis, dass die Flaeche (7) fuer den Fall, dass U eine Gewindekurve, also der Vector

$$(15) \quad U''' + kU' = c$$

eine konstante Richtung besitzt, eine geradlinige Asymptotenlinie w=konst. hat. Es wird naemlich in die Formel

$$\phi_u = U' + w(\alpha_u U' + \beta U''')$$

hier $$\alpha_u = -(U'''V'V'') = -(cV'V'') + k(U'V'V'')$$

$$= -(cV'V'') + k\beta$$

und der Wert von U''' aus (15) einzusetzen sein :

$$\phi_u = U'(1 - w(cV'V'')) + w\beta c,$$

woraus fuer $$w = \frac{1}{(cV'V'')}$$

ϕ_u sich in der Tat proportional dem Vektor c des Gewindedurchmessers ergibt.

Zur Vereinfachung der Rechnung setzen wir die Liesche Gleichung der Kurve U in der Form

$$(16) \quad U_1 U''_2 - U_2 U'_1 = m U'_3$$

an, so dass die Durchmesser des Gewindes parallel zur dritten Koordinatenachse werden, *d.h.* der Punkt c die Koordinaten o, o_1, c_3 besitzt.

Dann ist

$$(17) \quad U'''_1 + kU'_1 = 0$$

$$U'''_2 + kU'_2 = 0$$

$$U'''_3 + kU'_3 = c_3$$

waehrend durch mehrfache Ableitung sich aus (16)

$$(18) \quad U_1 U_2'' - U_2 U_1' = m U''_3$$

$$U'_1 U''_2 - U'_2 U''_1 = m c_3$$

ergibt. Damit kann man die Koordinaten von

$$\xi = [U'U'']$$

bezeichnen: Es wird

$$\xi_1 = U'_2U''_3 - U'_3U''_2 = \frac{1}{m}U_2(U'_1U''_2 - U'_2U''_1)$$

$$\xi_2 = U'_3U''_1 - U'_1U''_3 = -\frac{1}{m}U_1(U'_1U''_2 - U'_2U''_1)$$

waehrend ξ_3 aus der letzten Gleichung (18) gleich abgelesen werden kann. Damit wird (19)

$$\xi_1 = U_2c_3, \quad \xi_2 = -U_1c_3, \quad \xi_3 = mc_3.$$

Die Koordinaten der Achse sind damit:

$$\phi = U + V + \frac{1}{c_3(V'_1V''_2 - V'_2V''_1)}(\gamma U' + \delta VV''),$$

wobei $$\delta = (\xi V') = -c_3\{U_2V'_1 - U_1V'_2 + mV'_3\}$$

(19) $$\gamma = -\delta_v = c_3\{U_2V''_1 - U_1V''_2 + mV''_3\}$$

Setzt man diese Werte in die Formel fuer die Achse ϕ ein, so erhaelt mann fuer die Koordinaten ϕ_1 und ϕ_2 ihres Bildpunktes

$$\phi_1 = V_1 + \frac{m(V_2'V''_1 - V'_1V''_2)}{(V'_1V''_2 - V'_2V''_1)} = V_1 - m\frac{\eta_2}{\eta_3}$$

(20) $$\phi_2 = V_2 - \frac{m(V'_2V''_3 - V'_3V''_2)}{V'_1V''_2 - V'_2V''_1)} = V_2 + m\frac{\eta_1}{\eta_3}$$

Gehoert V einem koaxialen Gewinde an wie U, setzt man also

$$V_1V''_2 - V_2V'_1 = mV'_3$$

so ergibt eine entsprechende Rechnung

$$\eta_1 = V_2\bar{c}_3, \quad \eta_2 = -V_1\bar{c}_3, \quad \eta_3 = \bar{m}\,\bar{c}_3,$$

wobei $\bar{m}$ und $\bar{c}_3$ gewisse Konstanten bedeutet. Jedenfalls wird

$$\frac{\eta_1}{\eta_3} = \frac{V_2}{\bar{m}} \qquad \frac{\eta_2}{\eta_3} = -\frac{V_1}{\bar{m}}$$

so dass

$$\phi_1 = V_1\frac{\bar{m}+m}{\bar{m}} \qquad \phi_2 = V_2\frac{\bar{m}+m}{\bar{m}}$$

eine Kurve darstellt, die der Kurve, in die V projiziert wird, aehnlich ist.

ERRATA

Page 1.	Line 8.	*For*	"1·006"	*Read*	"1·0006."
Page 4.	Line 11.	*For*	"$N^{\frac{1}{2}}$"	*Read*	"$N^{-\frac{1}{2}}$"
Page 5.	Lines 15, 16.	*For*	"λ^3"	*Read*	"λ^2."
Page 5.	Line 17.	*For*	"10^{-7}"	*Read*	"10^{-11}."
Page 6.	Line 2,	*For*	"300"	*Read*	"170."
...	...	...	"400"	*Read*	"120."
Page 10.	Line 15.	*For*	"counter-clockwise"	*Read*	"clockwise."
Page 11.	Line 2.	*For*	"process"	*Read*	"powers."
Page 13.	Line 7.	*For*	"ρ_1"	*Read*	"ϕ_1."
Page 15.	Line 5.	*For*	"ρ"	*Read*	"ϕ."
Page 19.	Line 3.	*For*	"e^n"	*Read*	"*en*."
Page 19.	Line 6.	*For*	"361"	*Read*	"36 (1."
Page 19.	Line 15.	*For*	"$1+7\rho$"	*Read*	"$-1+7\rho$."
Page 19.	Line 17.	*For*	"Dim ale"	*Read*	"Decimale."
Page 32.	Line 9.	*For*	"dirrso"	*Read*	"avviso."
Page 33.	Line 16.	*For*	"(continua)."	*Read*	"(continua)nell' intervallo(o,l)."
Page 34.	Line 5.	*For*	"minore"	*Read*	"maggiore."
Page 36.	Line 7.	*For*	"$\phi_{n_{m_0}}^{x-\frac{1}{n_n}}\Big]$"	*Read*	"$\phi_{n_{m_0}}^{x-\frac{1}{n_n}}(y)\Big]$"
Page 39.	Line 15.	*For*	"lo"	*Read*	"la."
Page 42.	Line 5.	*For*	"vagano"	*Read*	"valgano"
Page 42.	Line 25.	*For*	"$\Psi(\lambda)$"	*Read*	"$\Psi(x)$."
Page 49.	Line 1.	*For*	"*Bersteinsche*"	*Read*	"*Bernsteinsche*'
Page 57.	Line 20.	*For*	" (4) "	*Read*	" (5)."
Page 59.	Last Line	*For*	" $bx+c$ "	*Read*	"$bxy+cy^2$."
Page 65.	Line 16.	*For*	" T^2-dU "	*Read*	" T^2-dU^2."
Page 67.	Line 2.	*For*	" *nétirages* "	*Read*	" *n tirages*."
Page 68.	Line 14.	*For*	" function "	*Read*	" fonction."
Page 68.	Line 28.	*For*	" c "	*Read*	" C."
Page 71.	Line 8.	*For*	" 1 t "	*Read*	" $1\leqq$t."
Page 155.	Line 7 of the foot-note.	*For*	"5"	*Read*	"25."
Page 161.	Line 11.	*For*	"left"	*Read*	"right."

Page 167. Line 3 from the bottom.	*For* "$\int_6^t$"	*Read* "$\int_0^t$"
Page 173. Last three lines.	*For* "$\frac{\chi}{\psi'} \prec v$ or $\frac{\chi}{\psi'} \succeq v$"	*Read* "$\psi \succ \log \frac{1}{v}$ or $\psi \preceq \log \frac{1}{v}$"
Page 174. Line 2.	*For* $\frac{\text{"}\chi\text{"}}{\psi'}$	*Read* $\frac{\text{"}1\text{"}}{\psi'}$.
Page 174. Line 3.	*For* "$0_{21}(t)$"	*Read* "$0_{21}(t). \frac{\chi(t)}{\psi'(t)}$"
Page 174. Line 4.	*For* "$0_{22}(h^2)$"	*Read* "$0_{22}(h^2). \frac{\chi(h)}{\psi'(h)}$"
Page 174. Line 9.	*For* "$\frac{0_{22}(h^2)}{2X_2(h)} =$"	*Read* "$\frac{0_{22}(h^2)}{2X_2(h)} . \frac{\chi(h)}{\psi'(h)} =$"
Page 174. Line 10.	*For* "$\frac{h0_{21}(h)}{2X_2(h)}$"	*Read* "$\frac{h0_{21}(h\theta)}{2X_2(h)} . \frac{\chi(h)}{\psi'(h\theta)}$"
Page 175. Line. 1.	*For* "$\frac{\chi}{\psi'} \prec h$"	*Read* "$\psi \succ \log \frac{1}{v}$"
Page 177. Line. 10.	*For* "$\prec h$"	*Read* "$\prec 1$."
Page 180. Lines 8, 10 of the foot-note.	*For* "Whitcomb"	*Read* "Whitcom"
Page 182. Line 4.	*For* "44"	*Read* "448."
Page 246. Line 1.	*For* "x"	*Read* "x_1"
Page 267. Line 6.	*For* "quadilaterals"	*Read* "quadrilaterals."
Page 268. Line 18.	*For* ".4"	*Read* "4."
Page 276. Line 13.	*For* "altitudes"	*Read* "altitude."
Page 279. Line 2.	*For* "Figs. 1 and 2"	*Read* "Figs. 3 and 4."
Page 288. Line 11.	*For* "$xy=nuv$"	*Read* "$xy=quv$."

ERRATA

Page 1.	Line 8.	*For*	"1·006"	*Read*	"1·0006."
Page 4.	Line 11.	*For*	"$N^{\frac{1}{2}}$"	*Read*	"$N^{-\frac{1}{2}}$"
Page 5.	Lines 15, 16.	*For*	"λ^3"	*Read*	"λ^2."
Page 5.	Line 17.	*For*	"10^{-7}"	*Read*	"10^{-11}."
Page 6.	Line 2.	*For*	"300"	*Read*	"170."
...	...	...	"400"	*Read*	"120."
Page 10.	Line 15.	*For*	"counter-clockwise"	*Read*	"clockwise."
Page 11.	Line 2	*For*	"process"	*Read*	"powers."
Page 13.	Line 7.	*For*	"ρ_1"	*Read*	"ϕ_1."
Page 15.	Line 5.	*For*	"ρ"	*Read*	"ϕ."
Page 19.	Line 3.	*For*	"e^n"	*Read*	"*en*."
Page 19.	Line 6.	*For*	"36 1"	*Read*	"36 (1."
Page 19.	Line 15.	*For*	"$1+7\rho$"	*Read*	"$-1+7\rho$."
Page 19.	Line 17.	*For*	"Dim ale"	*Read*	"Decimale."

APPENDIX.

THE TWENTIETH ANNIVERSARY OF THE FOUNDATION OF THE CALCUTTA MATHEMATICAL SOCIETY.

Proceedings of the Special Meeting held on the 28th December, 1928, at 4 p.m.

PRESENT :

PROFESSOR GANESH PRASAD, M.A., D.Sc., *President* of the Society, in the *Chair*.

PROFESSOR W. S. URQUHART, M.A., D.LITT., D.L., D.D., *Vice-Chancellor*, Calcutta University.

SIR DEVAPRASAD SARVADHIKARY, VIDYARATNAKAR, SURIRATNA, KT., C.I.E., C.B.E., M.A., LL.D.

PROFESSOR SIR P. C. RAY, KT., M.A., Sc.D., PH.D., C.I.E.

PRINCIPAL G. C. BOSE, M.A., Bangabasi College, Calcutta.

RAI BAHADUR J. C. GHOSH, M.A., *Registrar*, Calcutta University.

PROFESSOR SIR C. V. RAMAN, KT., F.R.S., Calcutta University.

PROFESSOR P. C. MITTER, M.A., PH.D., Calcutta University.

PROFESSOR S. K. MITTER, D.Sc., Calcutta University.

PROFESSOR J. N. MUKHERJEE, D.Sc., Calcutta University.

PROFESSOR P. N. GHOSH, M.A., PH.D., Sc.D., F.INST.P., Calcutta University.

REVEREND A. CAMERON, M.A., *Professor*, Scottish Churches College, Calcutta.

DR. B. DATTA, D.Sc., *Lecturer*, Calcutta University.

DR. S. M. GANGULY, D.Sc., *Lecturer*, Calcutta University.

DR. N. N. SEN, D.Sc., *Lecturer*, Calcutta University.

DR. A. B. DATTA, M.A., PH.D., *Professor*, Ripon College, Calcutta.

MR. S. C. GHOSH, M.A., *Lecturer*, Calcutta University.

MR. H. P. BANERJEE, M.Sc., *Lecturer*, Calcutta University.

DR. H. C. RAICHAUDHURI, M.A., PH.D., *Lecturer*, Calcutta University.

DR. B. N. CHUKERBUTTY, D.Sc., *Lecturer*, Calcutta University.

MR. S. N. MITRA, M.A., *Lecturer*, Calcutta University.

MR. S K. ACHARYA, M.Sc., *Lecturer*, Calcutta University.

MR. K. L VARMA, M.A , *Professor*, Maharaja's College, Jaipur.

MR. P. MATHUR, M.A , *Professor*, Maharaja's College, Jaipur.

DR. GORAKH PRASAD, D.Sc., *Reader in Mathematics*, Allahabad University.

DR. P. L. SRIVASTAVA, M.A., D PHIL., *Reader in Mathematics*, Allahabad University.

MR. B. N. PRASAD, M.Sc., *Lecturer in Mathematics*, Allahabad University.

MR. RAJKISHORE, M Sc., Government College, Ajmer.

MR. H. N. DATTA, M.Sc., *Reader in Mathematics*, Muslim University, Aligarh.

DR. S. C. MITRA, M.A , PH D , *Lecturer in Mathematics*, Dacca University.

MR. H. SIRCAR, M.Sc., *Lecturer in Mathematics*, Dacca University.

MR. B. SEN, M.Sc , *Professor of Mathematics*, Islamia College, Calcutta.

MR. A M. SEN, M.A., *Professor of Mathematics*, Presidency College, Calcutta.

MR. P. C. SEN GUPTA, M.A , *Professor of Mathematics*, Bethune College, Calcutta.

CAPTAIN H. DABIRUDDIN AHMAD, K.I.H., L.M.S., B.M.S., A.I.R.O. *Lecturer*, Campbell Medical School, Calcutta.

DR. G. N. BANERJI, M.A., PH.D , *Secretary*, Council of Post-Graduate Teaching in Arts, Calcutta University.

DR. B. B. DATTA, M.A., B.L., PH.D., *Assistant Controller of Examinations*, Calcutta University.

The proceedings began with an address from the President Prof. Ganesh Prasad, which appears on pages 101-108 of this volume.

After the President's speech, messages from a number of famous mathematicians all over the world were read by the Secretary of the Society. Some of the messages are the following:—

(1) From Prof **Sir Joseph Larmor** of the *Cambridge University*: "May I offer cordial congratulations to the Calcutta Mathematical Society on its twenty years of fruitful activity which has established its position among the main sources of mathematical science in the world. May the great age of Indian Mathematics be revived in our time."